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EVALUATION OF THE ALBEDO INTEGRAL FOR MARK I

Cord H. Link, Jr.

ARO, Inc.

February 1966

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NOTICES

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FOREWORD

The work reported herein was done at the request of Arnold Engineering Development Center (AEDC), Air Force Systems Command (AFSC), under Program Element 65402234.

The results of the research were obtained by ARO, Inc. (a subsidiary of Sverdrup and Parcel, Inc.), under Contract AF 40(600)-1200. The work was performed from February to August, 1964, under ARO Project No. SM3105, and the manuscript was submitted for publication on August 31, 1965.

This report is an extension of the work reported in AEDC-TDR-63-206 (February 1964).

This technical report has been reviewed and is approved.

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ABSTRACT

This report is concerned with the development of a fast computer method for evaluating the albedo integral. This integral defines the illumination on an arbitrarily oriented surface element at any point in space about a diffusely reflecting sphere. It enters the calculation of simulation control parameters in the Arnold Engineering Development Center Aerospace Environmental Chamber (Mark I). The semi-numerical method developed here is faster than ordinary numerical integration by a factor of about ten. A typical computer program, which formerly required about thirty minutes, now produces the same results in under four minutes.

CONTENTS

ILLUSTRATIONS

NOMENCLATURE

A_e	Albedo, fraction of solar radiation reflected by earth
C_γ	Horizon circle
C_θ	Terminator curve
C_ξ	"Target plane cut" curve
ΔA_1	Surface element on albedo source
ΔA_2	Target surface element
h	Altitude
I_s	Solar constant, intensity
I_2	Intensity of target illumination
L	Orbit angular momentum vector
N	Vector normal to target surface
N_s	Solar node vector
R	Target position vector
r_e	Radius of the albedo source sphere
S	Sun position (unit)vector
α	Azimuth angle
α_i	$(i = 1, 2 \cdots)$ Boundary values in azimuth
α_N	Azimuth of target normal vector
$\check{\alpha}$, $\hat{\alpha}$	Minimum, maximum values of azimuth angle
β	Nadir angle
β_N	Nadir angle of target normal vector
$\check{\beta}$, $\hat{\beta}$	Minimum, maximum values of nadir angle
γ	Relative altitude parameter, nadir angle of horizon
θ_e	Angular distance from S to arbitrary point (dA_1) on albedo sphere; source to sun view angle
θ_s	Angular distance from S to R
θ_v	Inclination of plane of N_s and R to S
ξ	Angle at dA_2 between N and dA_1; target view angle

ϕ_e	Angular distance of arbitrary source point dA_1 from R
ϕ_v	Orbital angular position, between N_s and R
ψ	Angle between normal at dA_1 and direction to dA_2; source view angle

SECTION I
INTRODUCTION

This report extends one of the problems discussed in an earlier report[1]: the development of a method for evaluating the "albedo integral". The aim of this study is to improve the speed at which certain quantities are computed for the control of simulation parameters in the Aerospace Environmental Chamber (Mark I). In earlier study programs, the albedo integral was evaluated by strictly numerical integration techniques. The present seminumerical method is faster, by nearly an order of magnitude, than the numerical methods formerly used. This method has been incorporated into a Fortran language computer subroutine.

A derivation of the albedo integral, for illumination intensity, is reproduced in Appendix I, under assumptions that the albedo source is a homogeneous sphere with a diffusely scattering (Lambert) surface, so that the albedo is otherwise independent of surface and atmospheric conditions.

SECTION II
THE ALBEDO PROBLEM

In order to properly control the simulation of secondary radiation (albedo and planet radiance) in Mark I, it is necessary to determine the illumination on an arbitrarily oriented surface element at arbitrary altitude and at any position in a trajectory or orbital flight near a reflecting celestial body.

A derivation of the albedo integral, which expresses the illumination intensity, is given in the previous report[2] under assumptions that the albedo source is a sphere having a homogeneous, diffusely scattering surface so that the albedo is otherwise independent of surface and atmospheric conditions. Then a different primary body may be distinguished

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[1]Cord H. Link, Jr. "Problems in Computing Radiation Control Functions for Mark I." AEDC-TDR-63-206, February 1964.

[2]Ibid.

by a solar constant suitable for the distance from the sun, its mean albedo, and its radius. This last factor enters all secondary illumination calculations since they depend on relative altitude.

$$\frac{I_2}{\Delta A_2}(\theta_s, \gamma, a_N, \beta_N) =$$

$$\int_{\hat{\alpha}(\theta_s, \gamma, a_N, \beta_N)}^{\hat{\alpha}(\theta_s, \gamma, a_N, \beta_N)} \int_{\hat{\beta}(\alpha; \theta_s, \gamma, a_N, \beta_N)}^{\hat{\beta}(\alpha; \theta_s, \gamma, a_N, \beta_N)} \cos\theta(\alpha, \beta; \theta_s, \gamma) \cos\xi(\alpha, \beta; a_N, \gamma_N) \sin\beta \, d\beta \, d\alpha$$

The integral contains four parameters which determine the configuration of boundaries of the surface over which the integration is to be carried out, namely, the albedo source region.

The limits of integration are functions of these same parameters as well as of the second integration variable. The parameters establish the limits for the second integration, as well as controlling the functional form of the integration limits.

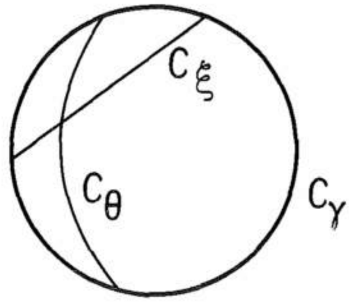
In principle either of the integration variables may be selected for the first integration. The azimuth angle α provides simple first integral forms, but the function limits, involving the nadir angle, are sometimes double valued functions, $\alpha(\beta)$.

On the other hand, the first integration taken relative to the nadir angle β leads to more complex expressions for the first integral, but the functional limits, involving the azimuth angle α, are single valued functions, $\beta(\alpha)$.

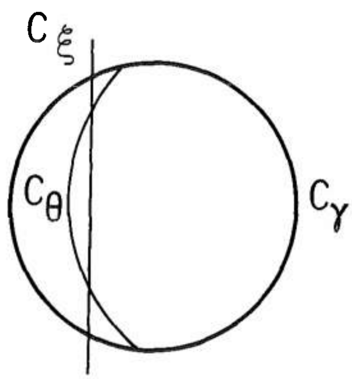
This second alternative is chosen. Having once found all the "antiderivatives" of the integrand functions of β, a simple differencing of function values for maximum and minimum values of β (at a particular value of α) provides first definite integral numerical values, which are now functions of the four configuration parameters and α. Integration over α involves summation of first definite integral values.

Regardless of which variable, α or β, is first used, when the function limits ($\alpha(\beta)$ or $\beta(\alpha)$) are inserted, some of the expressions become rather formidable, and analytical evaluations of the second

integrals for many of these have not been found. It is reasonable to use simple numerical integration in the remaining variable a.



The region of integration is bounded by curves beyond which one or more of the integrand factors become negative. There are three such curves (see sketch above). The ever-present horizon circle C_γ is determined by the relative altitude of the target above the albedo source. The terminator C_θ, the sunlight-shadow line, is determined both by altitude and by the angular distance of the target position from the subsolar point on the albedo source. Finally, the "target plane cut" C_ξ, the intersection of the plane of the target with the albedo source, depends on the specific orientation of the target and altitude. The curve C_θ may fall outside the circle, and the curve C_ξ does not exist outside the horizon circle. So, depending on the four parameters, the region of integration may be bounded by one curve C_γ, by two curves (C_γ with C_θ, C_θ with C_ξ, or C_γ with C_ξ), or finally by portions of all three curves.



Not only are the boundary curves defined by the four integration parameters, but their intersections are also, and there may be as many as six intersections (see above). From this arises part of the complexity of the problem, since the C_ξ curve may have any azimuthal relation to the C_θ curve, or within the C_γ circle. The logical sorting involved in determining the boundary curves and their limits, for arbitrary parameters, is rather involved in the number of decisions to be made. Yet for a given configuration, only one sequence of a few decisions serves to provide all the information required.

From the standpoint of computer programming, the method described here leads to a large program, of which only a small part is executed for one given set of parameters. In practice all the parameters may be continually varying.

In the following sections, the analysis will be developed, leading to the computer program displayed herein as a subroutine. A logical flow chart and Fortran II listing of the major routine is given as well as a Fortran II listing of the supporting subroutines. This method turns out to be approximately ten times as fast in computing as a corresponding purely numerical integration method.

SECTION III
THE ALBEDO INTEGRAL

The albedo integral, in its complete form, provides an expression for the intensity of illumination I_2 on an arbitrarily oriented and positioned target surface element ΔA_2 attributable to albedo A_e of a homogeneous diffusely scattering sphere exposed to solar radiation intensity I_s.[3] We begin with the definitions

$$\frac{I_2}{\Delta A_2} = \frac{I_s A_e}{\pi} \iint \cos\theta_e \cos\xi \sin\beta \, d\beta \, d\alpha \tag{1}$$

$$\cos\theta_e = \cos\theta_s \cos\phi_e + \sin\theta_s \sin\phi_e \cos\alpha \tag{2}$$

$$\cos\xi = \cos\beta \cos\beta_N + \sin\beta \sin\beta_N \cos(\alpha - \alpha_N) \tag{3}$$

$$\sin\gamma = r_e/(h + r_e) \tag{4}$$

$$\phi_e = \psi - \beta \tag{5}$$

$$\sin\psi = \sin\beta/\sin\gamma \tag{6}$$

The integration is over all α, β within the region where the integrand factors are all positive. The parameters α_N, β_N define the orientation of a target surface element to the particular albedo source configuration defined by θ_s, γ.

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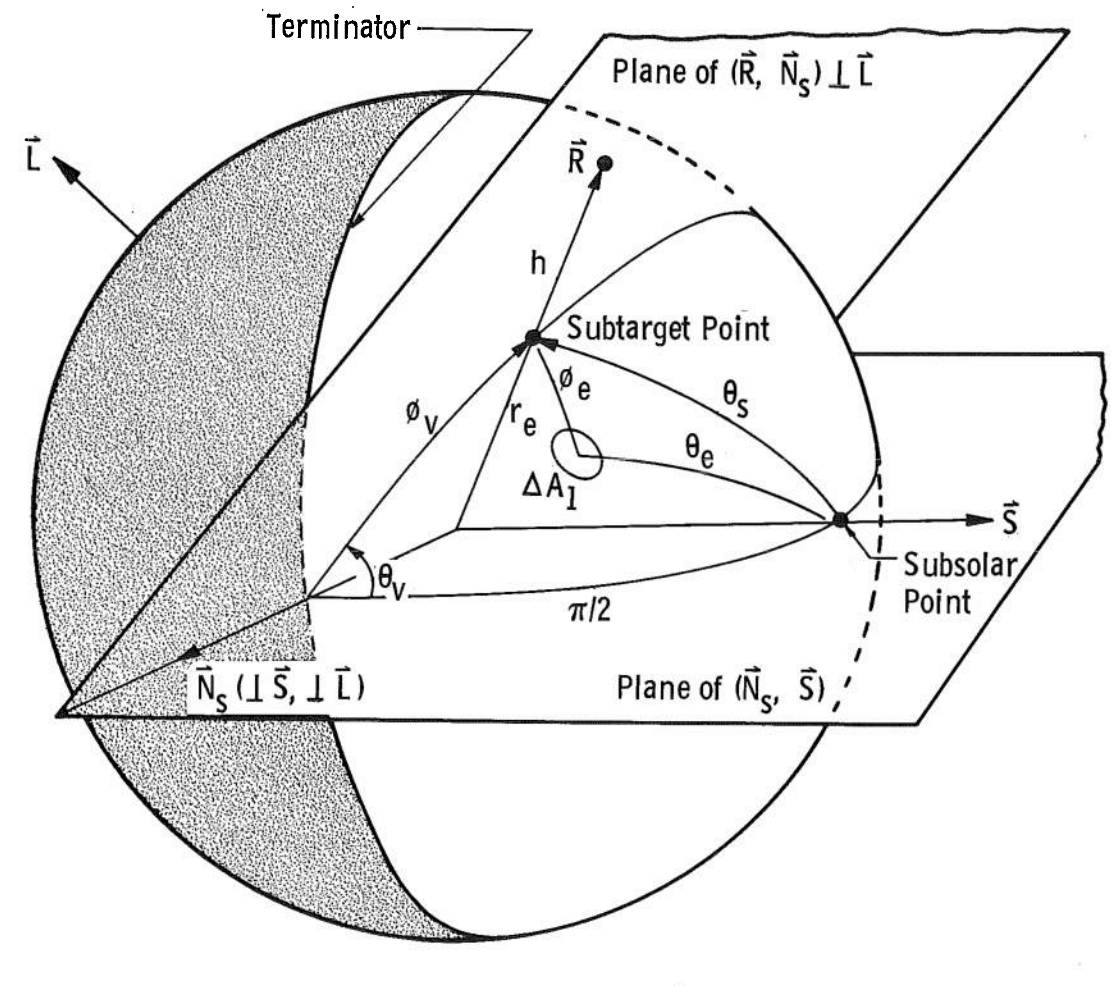
[3]Ibid.

For present purposes, the factor $I_s A_e$ is taken as unity, leaving the integral

$$\frac{1}{\pi} \iint \cos\theta_e \; \cos\xi \; \sin\beta \; d\beta \, d\alpha$$

which may be called the "albedo view factor". It is a measure of efficiency of conversion of collimated illumination into scattered illumination on an arbitrarily oriented surface element at any point in space about a perfect diffusely reflecting sphere.

The integration relative to nadir angle β is given in Appendix II.



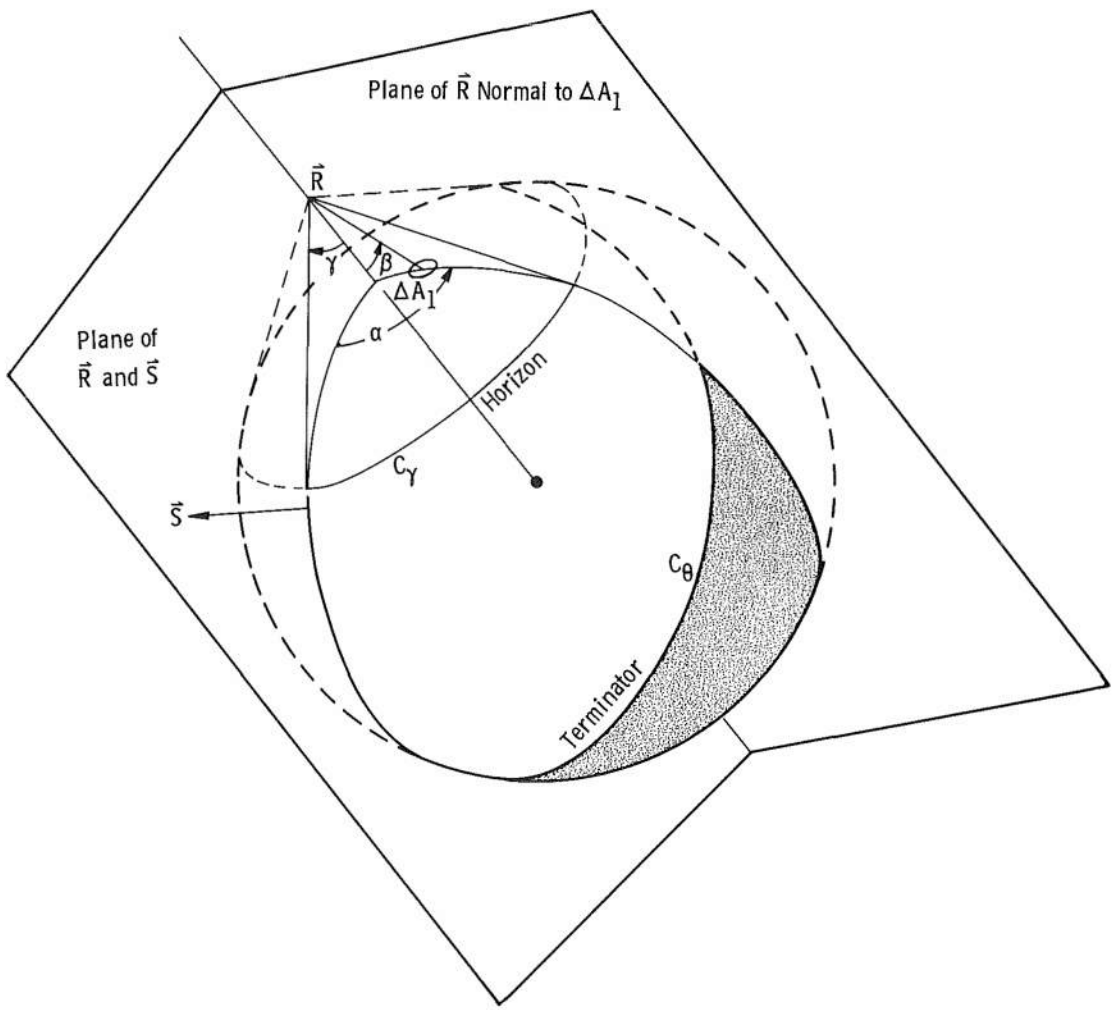
a. Geometry Defining Location Parameter $\theta_s(\theta_v, \phi_v)$

Fig. 1 Model Geometry

SECTION IV
PARAMETERS OF ALBEDO INTEGRAL

If S is a unit vector indicating the sun direction, a unit orbital angular momentum vector, and R the target position vector, then a node vector N_s may be constructed from S x L. Then the orbital angular position ϕ_v may be defined as the angle between N_s and R. The plane of the orbit is inclined at angle θ_v from the plane of N_s and S. Then the angular distance θ_s of R from S is defined by (Fig. 1a)

$$\cos \theta_s = \sin \phi_v \cos \theta_v$$



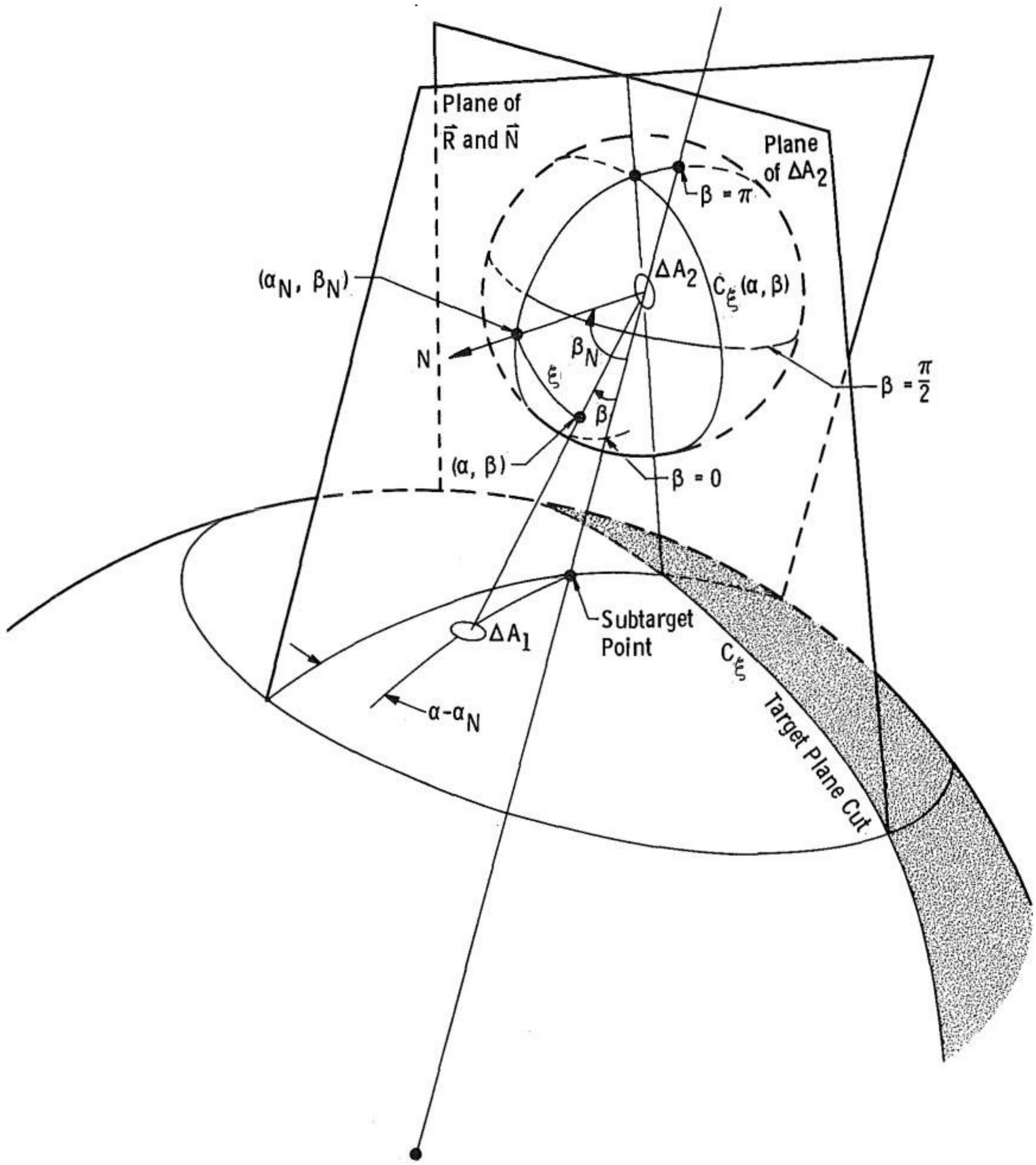
b. Geometry Defining α, β, γ, C_γ and C_θ

Fig. 1 Continued

The α, β coordinate system is defined at the vehicle position by polar coordinates, taking $\beta = 0$ as the $(-R)$ direction, $\alpha = 0$ in the plane of R and S, α positive by a right-hand rotation about $(+R)$, (Fig. 1b).

Then α_N, β_N are defined as the coordinates of the normal to the albedo target surface element ΔA_2 (Fig. 1c).

The relative altitude parameter γ is defined by Eq. (4). The relations of Eqs. (4) and (5) are shown in Fig. 1d.



c. Geometry Defining α_N, β_N, ξ and C_ξ

Fig. 1 Continued

SECTION V
BOUNDARY CURVES AND INTERSECTIONS

Since the boundary curves separate the region in (α, β) for which the integrand factors of Eq. (1) are positive from the region where any factor is negative, we may write boundary equations as follows:

C_γ (Horizon Circle): $\beta = \gamma$ (7)

for all α.

C_θ (Terminator, from Eq. (2)): $\cos \theta_e = 0$,

hence

$$\cos \alpha = -\operatorname{ctn} \phi_e \operatorname{ctn} \theta_s \tag{8}$$

in which Eqs. (4), (5), and (6) are used to obtain expressions for $\alpha(\beta)$ or $\beta(\alpha)$.

C_ξ (Target Plane Cut, from Eq. (3)):

$$\cos \xi = 0,$$

hence

$$\cos (\alpha - \alpha_N) = -\operatorname{ctn} \beta_N \operatorname{ctn} \beta \tag{9}$$

The intersections of C_ξ with C_γ are obtained from Eqs. (7) and (9) by letting $\Delta\alpha = \alpha - \alpha_N$.

Then

$$\cos \Delta\alpha = -\operatorname{ctn} \beta_N \operatorname{ctn} \gamma$$

$$\left.\begin{aligned} \alpha_1 &= \alpha_N + \Delta\alpha \\ \alpha_2 &= \alpha_N - \Delta\alpha \end{aligned}\right\} \text{(MODULO } 2\pi) \tag{10}$$

The intersection of C_θ with C_γ is found by using Eqs. (6), (5), and (7) with Eq. (8) as follows

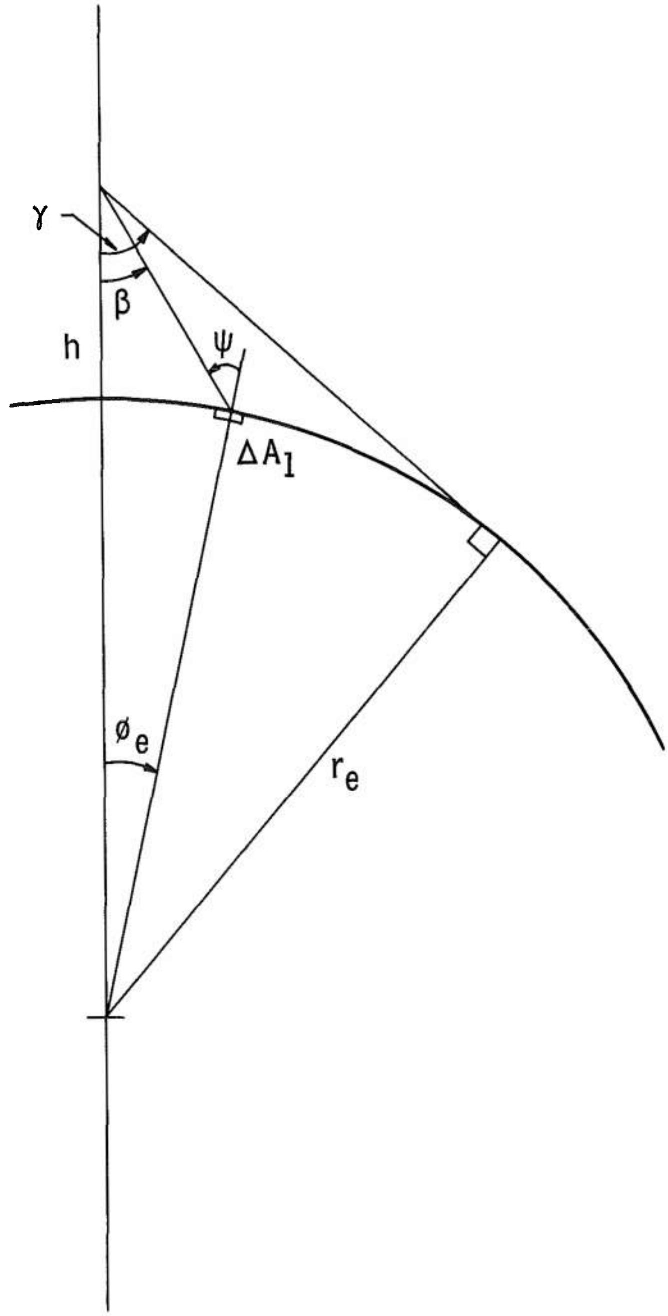
$$\beta = \gamma$$

$$\psi = \pi/2$$

$$\phi_e = \pi/2 - \gamma$$

$$\left.\begin{aligned} \alpha_3 &= \cos^{-1} (-\operatorname{ctn} \theta_s \tan \gamma) \\ \alpha_4 &= 2\pi - \alpha_3 \end{aligned}\right\} \tag{11}$$

The intersections of $C\theta$ with $C\xi$ are not needed in the present method, but will be essential if it is desired to attempt purely formal second-stage integration in the future. The calculation of this intersection is given in Appendix III; transformations between the angles ψ, ϕ_e, and B implied by Eqs. (5) and (6) are given in Appendix IV.



d. Geometry Defining Auxiliary Angles ϕ_e, ψ

Fig. 1 Concluded

It is desirable to ensure that all the angles of intersections (α_1, α_2, α_3, and α_4) of Eqs. (10) and (11) are expressed as positive angles within $(0, 2\pi)$ to eliminate ambiguities that otherwise occur in determining when the variable α is in the range of definition of one of the curves, C_θ or C_ξ.

It is apparent that the limit points α_1 and α_2 are symmetrically placed with respect to α_N at positions determined by γ and β_N. Correspondingly, α_3 and α_4 are symmetric relative to $\alpha = 0$ (located from expressions involving γ and θ_s). The four integration parameters γ, θ_s, α_N, β_N remain arbitrary, subject to limitations

$$0 \le \gamma \le \pi/2$$

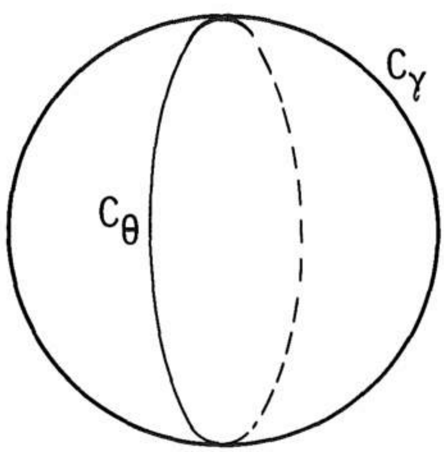
$$0 \le \theta_s \le \pi$$

$$0 \le \alpha_N \le 2\pi$$

$$0 \le \beta_N \le \pi$$

SECTION VI
MAJOR DIVISIONS OF PARAMETER RANGES

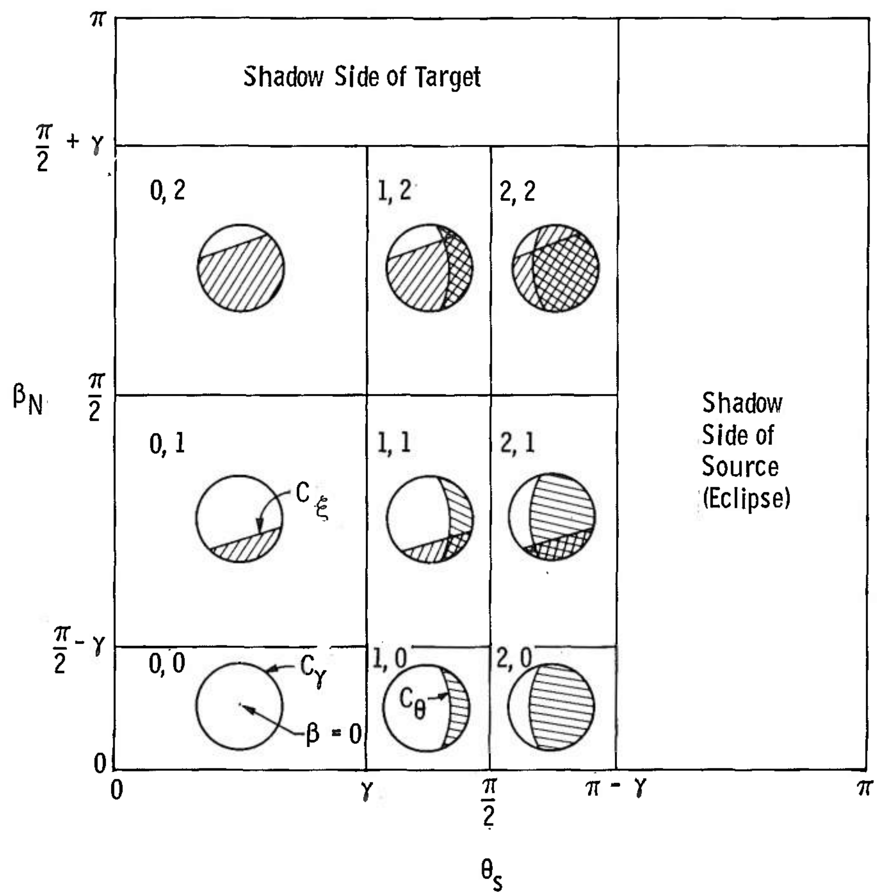
All required quantities are now defined, and we shall examine the meaning of the values of the four parameters α_N, β_N, θ_s, and γ. From the definition of γ (Eq. (4)), we find that γ approaches $\pi/2$ as the altitude vanishes, and γ approaches zero as altitude grows large.



(α, β) Map of C_θ in C_γ

We confine our attention to the horizon circle and its interior, $\beta \leq \gamma$. In the (α, β) coordinate system about the origin (the vehicle location), a unit radius sphere is erected. The horizon circle is a small circle, $\beta = \gamma$, which entirely encompasses the albedo source region, of which no part exists outside the horizon, $\beta > \gamma$. The curve C_ξ is a great circle on the α, β sphere and hence passes through the origin and maps across the interior of the horizon circle as a straight line. The curve C_θ, a great circle on the albedo source sphere, maps into the horizon circle as a part of an ellipse tangent to the horizon circle. The values of θ_s and β_N control the existence of C_θ and C_ξ within C_γ and the points of closest approach of these curves to $\beta = 0$. The missing ellipse branch of the C_θ curve, the continuation of the terminator, is not defined in (α, β), since it is physically outside the horizon circle or "behind" it (see sketch on page 10).

Figure 2a illustrates the major divisions of characteristics imposed by θ_s, β_N, and γ and by typical patterns of the integration region. For this illustration, α_N is arbitrarily set at $\pi/2$. Later, we shall examine the influence of α_N on the problem. Figure 2a illustrates schematically some typical boundary patterns.



a. Typical Patterns for $\gamma = \pi/3$

Fig. 2 Major Divisions of Parameter Ranges in the Horizon Circle C_γ

The major ranges are noted in Fig. 2a by use of paired numbers (n_1, n_2), the first referring to the θ_s range, the second to the β_N range. In range (0, 0), only C_γ bounds the region, and for integration we have $0 \leq \beta \leq \gamma$, $0 \leq \alpha \leq 2\pi$. This corresponds to a point on the vehicle nearest the source sphere, near $\beta = 0$ and a location of the vehicle not far from the subsolar point ($\theta_s = 0$) on the albedo source sphere.

With no other changes, as β_N increases we move from (0, 0) to (0, 1) where C_ξ comes into the horizon circle. The vehicle itself begins to mask part of the source. The point $\beta = 0$ is still within the source so the α limits are 0 and 2π; $\beta = 0$ is the minimum β value ($\check{\beta}$), and the maximum ($\hat{\beta}$) is either γ or dependent on α through the equation for curve C_ξ.

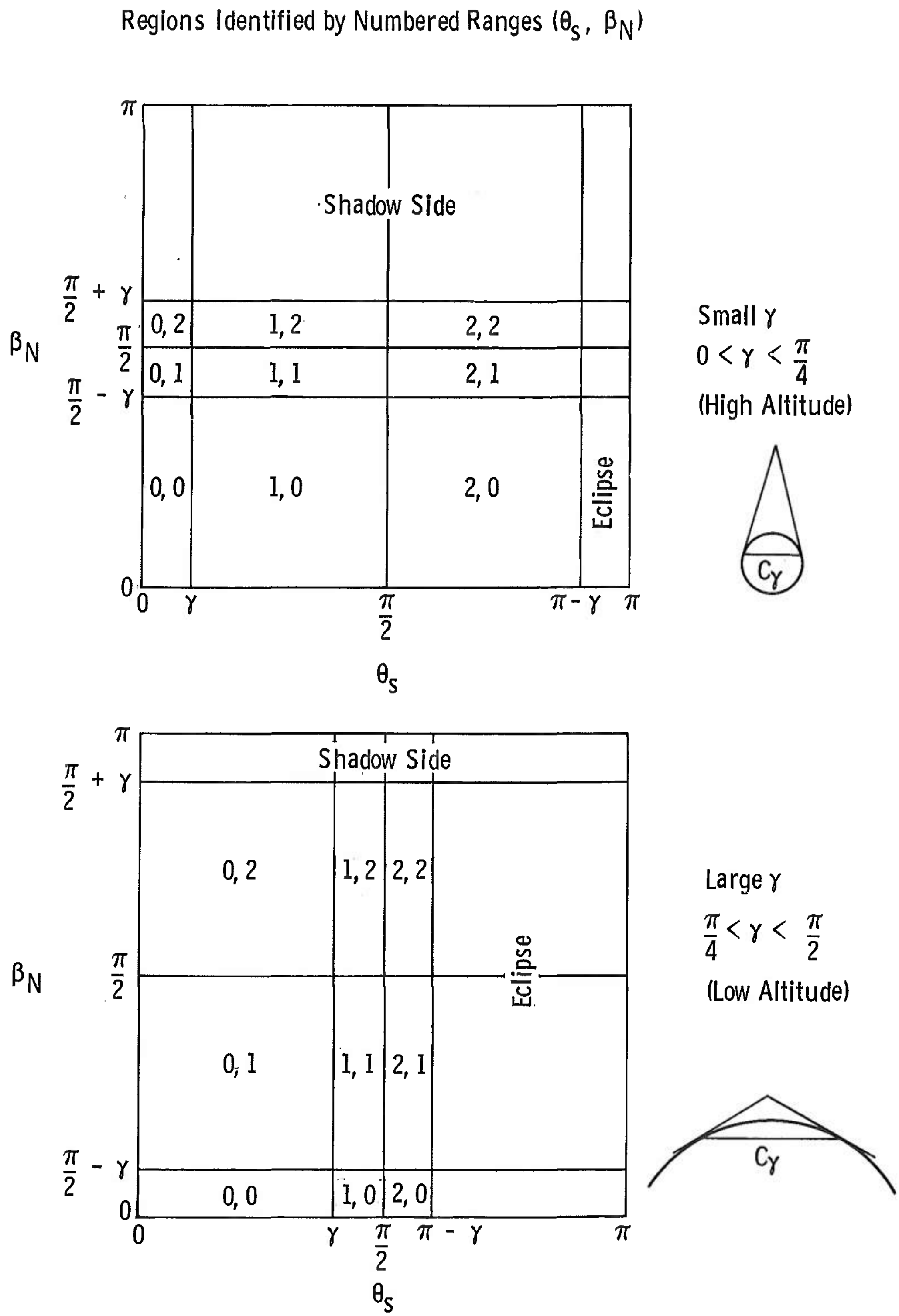
When $\beta_N = \pi/2$, the target plane (or its equivalent C_ξ) bisects the horizon circle. Now α has a range of $\pi/2$ either side of α_N, $\check{\beta}$ is zero, and $\hat{\beta}$ is γ only. Or we may allow α its full $(0, 2\pi)$ range, but during half of this range the curve C_ξ provides $\hat{\beta} = 0$ and in the other half C_γ gives $\hat{\beta} = \gamma$ while $\check{\beta} = 0$.

As β_N grows, C_ξ moves into (0, 2), on past the nadir point $\beta = 0$, and provides $\check{\beta}$ while $\hat{\beta}$ is γ. The range of α is now (α_1, α_2), the region where C_ξ is defined. Finally, β_N increases so far that it is "on top" of the vehicle, C_ξ has swept completely across the interior of the horizon circle, and the integral value becomes zero. The target now completely masks itself from the albedo source.

It must be noted that the integrand contains $\cos \theta_e$, θ_e being measured from the subsolar point on the albedo sphere. Thus, generally, the source intensity is not symmetric in any way unless $\beta_N = 0$ or π, and these two instances are not equivalent since one of them includes areas nearer the subsolar point, and the other is directly opposite. But α_N is arbitrary, in practice a function of β_N determined by the vehicle geometry and orientation.

The dependence of the θ_s and β_N ranges on the altitude parameter, γ, is illustrated in Fig. 2b.

We now return to case (0, 0) and allow θ_s to vary. As we enter (1, 0), the terminator C_θ appears in the horizon circle at $\alpha = \pi$. The nadir point is still within the integration region so α ranges $(0, 2\pi)$, $\check{\beta} = 0$, and β are determined from either C_θ or C.



b. Effect of γ on Boundaries

Fig. 2 Concluded

Allowing θ_s to increase to $\pi/2$ (as the vehicle crosses over the terminator), $C\theta$ bisects the source field, and we move on into (2, 0). Here $C\theta$ provides $\check{\beta}$, $\hat{\beta} = \gamma$, and α ranges only over α_3, α_4.

Finally θ_s increases so far that $C\theta$ leaves the horizon circle at $\alpha = 0$, and we have the eclipse condition where no part of the illuminated albedo sphere is visible at the vehicle. The integral vanishes. We note that cos ξ in the integrand also destroys the apparent symmetry in α of the source function except in special cases. Instances where symmetry occurs were treated in Appendix II of the previous report[4] as special cases in which the albedo integral can be obtained in closed form.

The parameter ranges labeled (1, 1), (1, 2), (2, 1), and (2, 2) are superpositions of those just described. The bounds are dependent on all four parameters. When only one of the curves $C\xi$ or $C\theta$ establishes the β, then the α range necessarily lies within the corresponding end points (α_1, α_2) or (α_3, α_4). More precise statements are developed in subsequent sections as we go more deeply into the logic of sorting out the various cases.

SECTION VII
CONFIGURATIONS OF ALBEDO SOURCE BOUNDARIES

In this section, we display 43 distinct configurations of boundaries covering all useful values of the four integration parameters. All of these must be examined for the purpose of establishing exact integration ranges in α, ranges in which the boundaries are different functions $\beta(\alpha)$. As earlier indicated, we shall eventually integrate over (α, β) by using exact expressions for the first definite integral in β, which contains functions $\check{\beta}(\alpha)$ and $\hat{\beta}(\alpha)$, then numerically integrating in α.

We recall that all end points (α_1, α_2, α_3, α_4) are defined to have values in $(0, 2\pi)$, that the curve $C\gamma$ is defined by $\beta = \gamma$ for all α, that $C\xi$ is defined (Eq. (8)) in (α_1, α_2), and $C\theta$ is defined in (α_3, α_4). We do not require the intercepts of $C\theta$ with $C\xi$, which would be denoted (α_5, α_6), because results based on this knowledge are readily obtainable by an artifice which we shall use in the numerical α integration. These points would be

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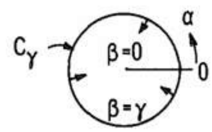
[4] Ibid.

required if one were to attempt to find a complete analytical expression for the albedo integral.

We return to the notation of the previous section for discussion of the major divisions. Viewed from the origin of the (α, β) coordinate system, boundary curve C_γ is a circle whose interior contains the regions of interest. Curve C_ξ is a straight line segment, and C_θ, although an ellipse section tangent to C_γ, is indicated as a circular arc for clarity and ease of drawing. The variously numbered points are simply numbered in sketches. The sides of C_ξ and C_θ on which the corresponding integrand factors are positive are indicated by a small arrow, pointing toward α_N on C_ξ, and toward $\alpha = 0$ on C_θ, or to the "interior" of these curves. Then the region of interest is just that part of the pattern which is common to the interiors of all three curves.

Case (0, 0): $\beta_N \leq \frac{\pi}{2} - \gamma$, $\theta_s \leq \gamma$

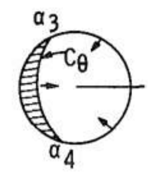
$0 \leq \alpha \leq 2\pi$, $\check{\beta} = 0$, $\hat{\beta} = \gamma$



Case (1, 0): $\beta_N \leq \frac{\pi}{2} - \gamma$, $\gamma \leq \theta_s \leq \frac{\pi}{2}$

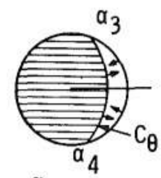
$0 \leq \alpha \leq 2\pi$ $\quad \check{\beta} = 0$ $\quad \hat{\beta} = \gamma$

or $\hat{\beta} = \beta(C_\theta)$ for $\alpha_3 \leq \alpha \leq \alpha_4$



Case (2, 0): $\beta_N \leq \frac{\pi}{2} - \gamma$ $\quad \frac{\pi}{2} \leq \theta_s \leq \pi - \gamma$

$\left.\begin{array}{l} 0 \leq \alpha \leq \alpha_3 \\ \text{and } \alpha_4 \leq \alpha \leq 2\pi \end{array}\right\}$ split scan $\quad \check{\beta} = \beta(C_\theta)$ $\quad \hat{\beta} = \gamma$



Case (0, 1): $\theta_s \leq \gamma$ $\quad \frac{\pi}{2} - \gamma \leq \beta_N \leq \frac{\pi}{2}$

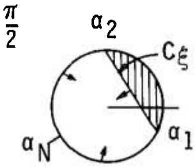
$0 \leq \alpha \leq 2\pi$

$\check{\beta} = 0$

$\hat{\beta} = \gamma$

or $\hat{\beta} = \beta(C_\xi)$

if $\left\{\begin{array}{l} 0 \leq \alpha \leq \alpha_2 \\ \alpha_1 \leq \alpha \leq 2\pi \end{array}\right\}$ split test



$0 \leq \alpha \leq 2\pi$

$\check{\beta} = 0$

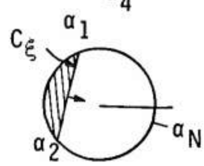
$\hat{\beta} = \gamma$

or

$\hat{\beta} = \beta(C_\xi)$

if

$\alpha_1 < \alpha < \alpha_2$

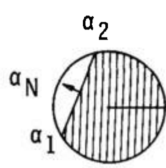


Case (0, 2): $\theta_s \leq \gamma$ $\quad \frac{\pi}{2} \leq \beta_N \leq \frac{\pi}{2} + \gamma$

$\hat{\beta} = \gamma$

$\check{\beta} = \beta(C_\xi)$

$\alpha_2 \leq \alpha \leq \alpha_1$



$\hat{\beta} = \gamma$

$\check{\beta} = \beta(C_\xi)$

$\left.\begin{array}{l} 0 \leq \alpha < \alpha_1 \\ \text{and } \alpha_2 \leq \alpha \leq 2\pi \end{array}\right\}$ split scan

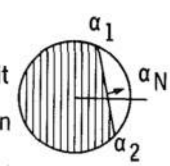


Fig. 3 Configurations of One and Two Boundary Curves

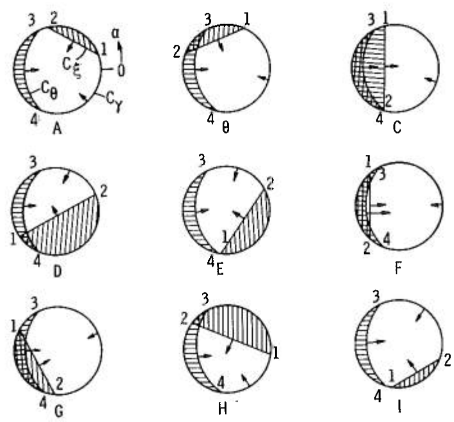
Cases (0, 0), (0, 1), (0, 2), (1, 0), and (2, 0), shown in Fig. 3, are largely self-explanatory. Cases (0, 1), (0, 2), and (2, 0), however, introduce the problem of the "split range". Although the two orientations shown in (0, 1) and (0, 2) are geometrically equivalent, they are logically distinct since all intersection values are defined in $(0, 2\pi)$. For example, in case (1, 0) when the order of end points is $\alpha_1 < \alpha_2$, the $C\xi$ curve is defined in (α_1, α_2), but when the order is $\alpha_2 < \alpha_1$, $C\xi$ is defined in the split range $(0, \alpha_2)$ and $(\alpha_1, 2\pi)$. Thus, the order of end points is essential to the orderly determination of the range in which α may be during numerical integration and for the selection of the proper boundaries $(\check{\beta}, \hat{\beta})$ for a given α.

The split range is also used in the initiation and advance of α during numerical integration. If the range is split, α is scanned over the two parts successively.

Case (1, 1) is the most complicated group of configurations because the α range is $0, 2\pi$ and both $C\xi$ and $C\theta$ are present, Fig. 4. Each configuration is labeled by letter referring to the corresponding permutation

Three Boundary Curves

$\frac{\pi}{2} - \gamma \leq \beta_N \leq \frac{\pi}{2}$, $\gamma \leq \theta_S \leq \frac{\pi}{2}$, $\check{\beta} = 0$, $0 \leq \alpha \leq 2\pi$



Configuration	α Order	Boundary Curve for $\hat{\beta}$					Notes
A	(0) 1 2 3 4 (2π)	γ	ξ	γ	θ	γ	
B	1 3 2 4	γ	ξ	(θ, ξ)	θ	γ	
C	1 3 4 2	γ	ξ	ξ	ξ	γ	
D	2 3 1 4	ξ	γ	θ	(θ, ξ)	ξ	*
E	2 3 4 1	ξ	γ	θ	γ	ξ	*
F	3 1 2 4	γ	θ	(θ, ξ)	θ	γ	
G	3 1 4 2	γ	θ	(θ, ξ)	ξ	γ	
H	3 2 4 1	ξ	(θ, ξ)	θ	γ	ξ	*
I	3 4 1 2	γ	θ	γ	ξ	γ	

Symbol	Meaning
(θ, ξ)	$\hat{\beta} = \mathrm{Min}(\beta(C_\theta), \beta(C_\xi))$
γ	$\hat{\beta} = \gamma$
θ	$\hat{\beta} = \beta(C_\theta)$
ξ	$\hat{\beta} = \beta(C_\xi)$
*	Split Test

Fig. 4 Case (1, 1)

of end points, given in the table at the bottom of the figure. Beside each permutation appears the order of subscripts in the boundary curves from which $\hat{\beta}$ is to be found. For this case, the point $\beta = 0$, lying interior to the integration region, is also $\check{\beta} = 0$. Where a double subscript occurs, we make use of the artifice (previously mentioned) to determine whether to use C_θ or C_ξ. Here, when $\hat{\beta}$ is to be found in a range of α where both C_θ and C_ξ are defined, we use the C_θ and C_ξ definitions to determine both values of $\hat{\beta}$, i.e., $\hat{\beta}(C_\theta)$ and $\hat{\beta}(C_\xi)$; we then select the least of these to be $\hat{\beta}$.

We give an example of interpretation of the table of Fig. 4. Select configuration B. Initiate α at 0, change the value by the (fixed) step size, and test a to see when it is in each subsequent range. In $(0, \alpha_1)$, $\hat{\beta} = \gamma$; in (α_1, α_3), $\hat{\beta} = \beta(C_\xi)$. In (α_3, α_2), $\hat{\beta} = \min[\beta(C_\theta), \beta(C_\xi)]$; then in (α_2, α_4), $\hat{\beta} = \beta(C_\theta)$. Finally, in $(\alpha_4, 2\pi)$, $\hat{\beta} = \gamma$.

Note that in any configuration in which $\alpha_2 < \alpha_1$, the range test must be split, as noted earlier.

Case (1, 2) in Fig. 5 has $\check{\beta}$ defined by the curve C_ξ, and, hence, α is scanned only over the range of definition of C_ξ; that is, over (α_1, α_2) if $\alpha_1 < \alpha_2$, or over $(0, \alpha_2)$ and $(\alpha_1, 2\pi)$ if $\alpha_2 < \alpha_1$. In the table, the symbol $[\theta]$ means that we use $\hat{\beta} = \beta(C_\theta)$ only if it is greater than $\check{\beta} = \beta(C_\xi)$; otherwise, there is no contribution to the integral for the current value of α.

Case (2, 1) in Fig. 6 has $\check{\beta}$ defined by C_θ, so α is scanned over $(0, \alpha_3)$ and $(\alpha_4, 2\pi)$, a split scan. The notation ξ^* in the table means that $\hat{\beta} = \beta(C_\xi)$ if only $\hat{\beta} > \check{\beta}$; otherwise, the current value at α contributes nothing to the integral.

Finally, case (2, 2) in Fig. 7 has $\hat{\beta}$ defined by $\beta = \gamma$, and we select $\check{\beta}$ as the greatest of $\beta(C_\xi)$ and $\beta(C_\theta)$, which is the meaning of the symbol (θ, ξ) in the table. Note that configuration G illustrates that nonoverlapping of the ranges of (α_1, α_2) and (α_3, α_4) leads to zero value of the integral. For case (2, 2) we let α scan only the least of the spans of C_θ or C_ξ; if this is C_θ then the α scan is split, but if C_ξ, the α scan may or may not be split. Notations of split scan test appear on the figures.

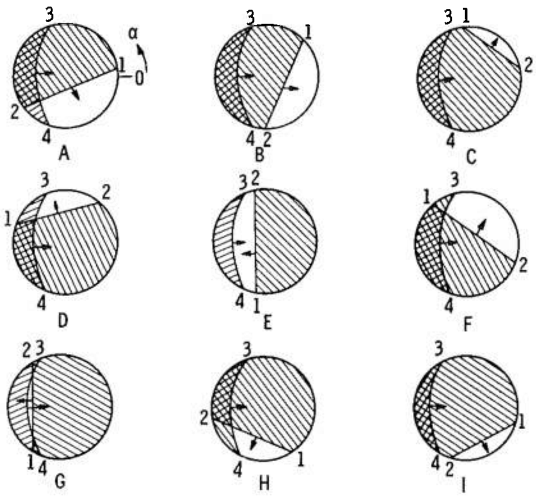
This completes the details of the logical procedures for doing the numerical integration in α. From the tables on the figures, the logic flow chart in Appendix V was derived; the problem was then programmed for computer directly from the flow chart. The Fortran listing is shown in Appendix VI.

Three Boundary Curves

$\frac{\pi}{2} < \beta_N < \frac{\pi}{2} + \gamma$, $\gamma \le \theta_s \le \frac{\pi}{2}$, $\check{\beta} = \beta(C_\xi)$

if $\alpha_2 < \alpha_1$, then $\alpha_2 \le \alpha \le \alpha_1$; but if $\alpha_1 < \alpha_2$,

then $0 \le \alpha \le \alpha_1$ and $\alpha_2 \le \alpha \le 2\pi$



Configuration	α Order	Boundary Curves for $\hat{\beta}$					Notes
A	(0) 1 3 2 4 (2π)	γ	-	-	[θ]	γ	*
B	1 3 4 2	γ	-	-	-	γ	*
C	2 1 3 4	-	γ	-	-	-	
D	2 3 1 4	-	γ	[θ]	-	-	
E	2 3 4 1	-	γ	θ	γ	-	
F	3 1 4 2	γ	[θ]	-	-	γ	*
G	3 2 1 4	-	-	[θ]	-	-	
H	3 2 4 1	-	-	[θ]	γ	-	
I	3 4 2 1	-	-	-	γ	-	

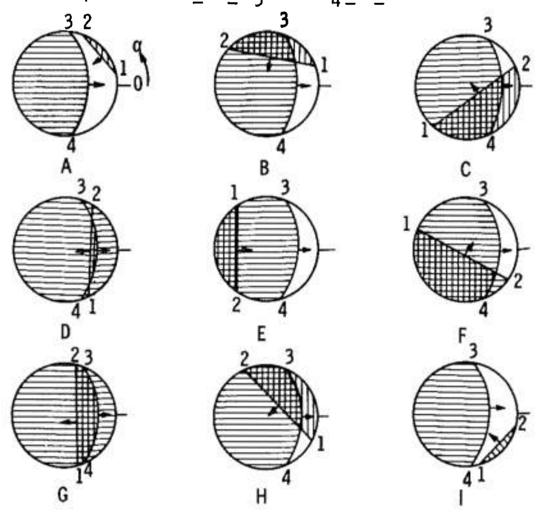
Symbol	Meaning
[θ]	$\hat{\beta} = \beta(C_\theta)$ only if $\hat{\beta} > \check{\beta}$, Otherwise No Contribution to Integral
-	Means No Contribution to Integral
*	Split Scan in α

Fig. 5 Case (1, 2)

Three Boundary Curves

$\frac{\pi}{2} - \gamma \le \beta_N \le \frac{\pi}{2}$, $\frac{\pi}{2} \le \theta_s \le \pi - \gamma$, $\check{\beta} = \beta(C_\theta)$

Split Scan $0 \le \alpha \le \alpha_3$ and $\alpha_4 \le \alpha \le 2\pi$



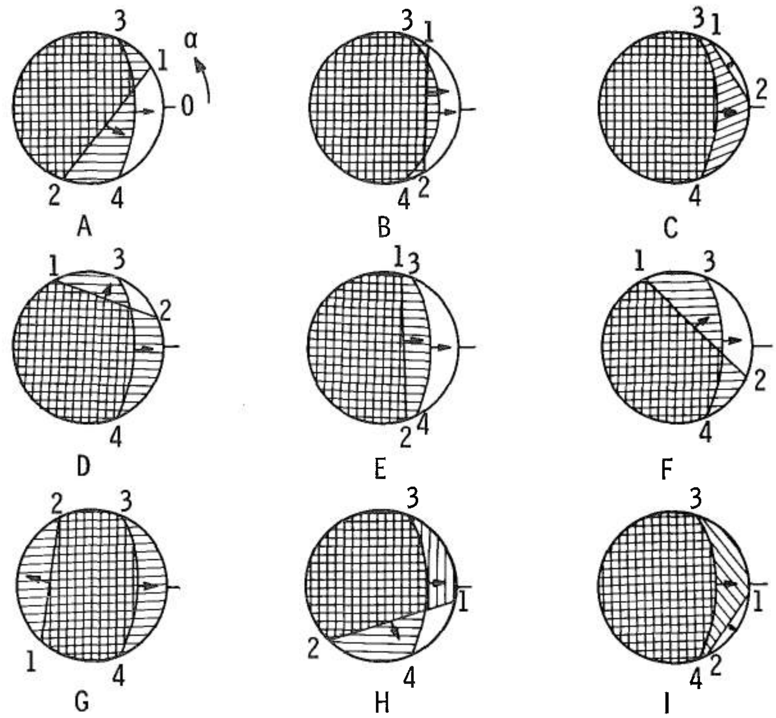
Configuration	α Order	Boundary Curves for $\hat{\beta}$					Notes
A	(0) 1 2 3 4 (2π)	γ	ξ*	γ	-	γ	*
B	1 3 2 4	γ	ξ*	-	-	γ	*
C	2 3 1 4	ξ*	γ	-	-	ξ*	
D	2 3 4 1	ξ*	γ	-	γ	ξ*	*
E	3 1 2 4	γ	-	-	-	γ	*
F	3 1 4 2	γ	-	-	ξ*	γ	*
G	3 2 1 4	-	-	-	-	-	**
H	3 2 4 1	ξ*	-	-	γ	ξ*	
I	3 4 1 2	γ	-	γ	ξ*	γ	*

Symbol	Meaning
ξ*	$\hat{\beta} = \beta(C_\xi)$ if $\hat{\beta} > \check{\beta}$ Otherwise No Contribution
*	Split Scan
**	Identically Zero Integral

Fig. 6 Case (2,1)

Three Boundary Curves $\frac{\pi}{2} < \beta_N < \frac{\pi}{2} + \gamma$, $\frac{\pi}{2} \leq \theta_s \leq \pi - \gamma$, $\hat{\beta} = \gamma$

α Ranges over Least Span, Covering Range of C_ξ if $\Delta\alpha \leq \alpha_3$, Otherwise over C_θ. Span of C_θ Always Split $0 \leq \alpha \leq \alpha_3$ and $\alpha_4 \leq \alpha \leq 2\pi$



Configuration	α Order	Boundary Curves for $\check{\beta}$					Notes
A	(0) 1 3 2 4 (2π)	(θ, ξ)	-	-	-	(θ, ξ)	*
B	1 3 4 2	(θ, ξ)	-	-	-	(θ, ξ)	*
C	2 1 3 4	-	(θ ξ)	-	-	-	
D	2 3 1 4	-	(θ, ξ)	-	-	-	
E	3 1 2 4	θ	-	-	-	θ	*
F	3 1 4 2	(θ, ξ)	-	-	-	(θ, ξ)	*
G	3 2 1 4	-	-	-	-	-	**
H	3 2 4 1	-	-	-	(θ, ξ)	-	
I	3 4 2 1	-	-	-	(θ, ξ)	-	

Symbol	Meaning
(θ, ξ)	$\check{\beta}$ = Max [$\beta(C_\theta)$, $\beta(C_\xi)$]
*	Split Test if α Ranges over C_ξ
**	Identically Zero Integral

Fig. 7 Case (2, 2)

SECTION VIII
CONCLUSIONS

In an attempt to gain computing speed in the evaluation of the albedo integral, a double integral, the problem has been changed from a straightforward numerical integration to a much faster but more complex semi-numerical integration. For example, whereas formerly the α, β range was covered by a mesh of 72 X 36 points, the present method requires somewhat more computation per point through a more complex logic network at only 72 points, and the accuracy is improved by the formal first integration. The time improvement is approximately one order of magnitude.

In application, a particular Mark I control program formerly required from 25 to 35 min (IBM 7074) to generate simulation parameters for a 90-min orbit with a simulation interval of two minutes. The same results are now produced in approximately four minutes.

It appears unlikely that significant gains in computing speeds can be made by using a purely formal solution to this problem. Second integrals will contain many more terms, some quite complex, and much of the gains made by having a single evaluation to perform will be lost in the sheer bulk of the expressions involved. Most of the logic of the present method would still apply for selecting integration limits and function groups to be evaluated. Some gain may result in changing the variable of first integration, and this will be studied in the future. It may also be possible to develop rapidly computing empirical approximating functions, especially over limited ranges of the integral parameters.

APPENDIX I
DERIVATION OF THE ALBEDO INTEGRAL

In the following discussion, the albedo source body is taken to be the earth. Substitution of appropriate values for radius, albedo, and solar constant allows extension to any source body.

To compute earth albedo and radiance integrals for a surface element having arbitrary orientation and position, we make the following assumptions:

1. that Albedo is a uniform property of the earth's surface,
2. that the earth is a sphere,
3. that the earth's surface is diffusely reflecting, and
4. that the earth has no atmosphere.

I_s Solar constant, intensity of solar radiation at earth

A_e Albedo, fraction of solar constant reflected, a surface property.

The solar radiation incident on an area element ΔA_1 having its normal inclined at angle θ_e to sun direction is

$$\Delta I_e = \begin{cases} I_s \cos \theta_e \; \Delta A_1 & \text{for} \quad \cos\theta_e \geq 0 \\ 0 & \text{for} \quad \cos\theta_e < 0 \end{cases}$$

Of this a fraction A_e is reflected diffusely by ΔA_1; hence, the intensity per unit solid angle ΔI_ψ in a direction inclined at angle ψ to the surface normal is

$$\Delta I_\psi = \frac{A_e \; I_s}{\pi} \cos\theta_e \; \cos\psi \; \Delta A_1$$

The intensity included in solid angle $\Delta\omega$ is

$$\Delta I_\omega = \frac{A_e \; I_s}{\pi} \cos\theta_e \; \cos\psi \; \Delta A_1 \; \Delta\omega$$

An area element ΔA_2, at distance ρ_e from ΔA_1, having its normal inclined at angle ξ to the direction of ρ_e, intercepts a solid angle (Fig. I-1).

$$\Delta\omega = \frac{\Delta A_2 \; \cos\xi}{\rho_e^2}$$

Hence the intensity arriving at ΔA_2 is

$$\Delta I_2 = \frac{A_e \ I_s}{\pi \ \rho_e^2} \cos\theta_e \ \cos\psi \ \cos\xi \ \Delta A_1 \ \Delta A_2$$

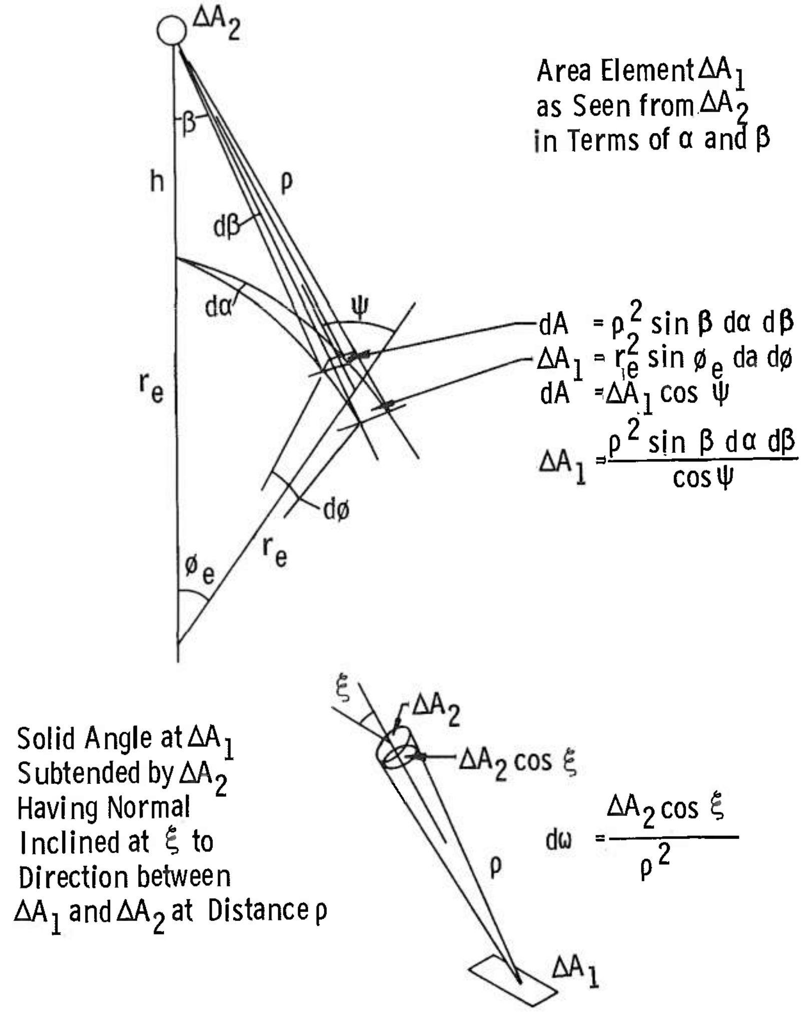


Fig. I-1 Solid Angle Geometry for Albedo and Earth Radiance Calculation

Let the area element ΔA_2 be located at altitude h above earth of radius r_e. From this point, the portion of the earth that can be seen is confined within a horizon circle. The angle γ between the direction of earth center and the horizon circle is defined by

$$\sin\gamma = r_e / (r_e + h) \qquad \left(0 < \gamma < \frac{\pi}{2}\right)$$

At the earth center, let θ_s be the angle between the direction to ΔA_2 and the sun direction; let θ_e be the angle between the area element ΔA_1 on earth and the sun; let ϕ_e be the angle between the directions of ΔA_1 and ΔA_2.

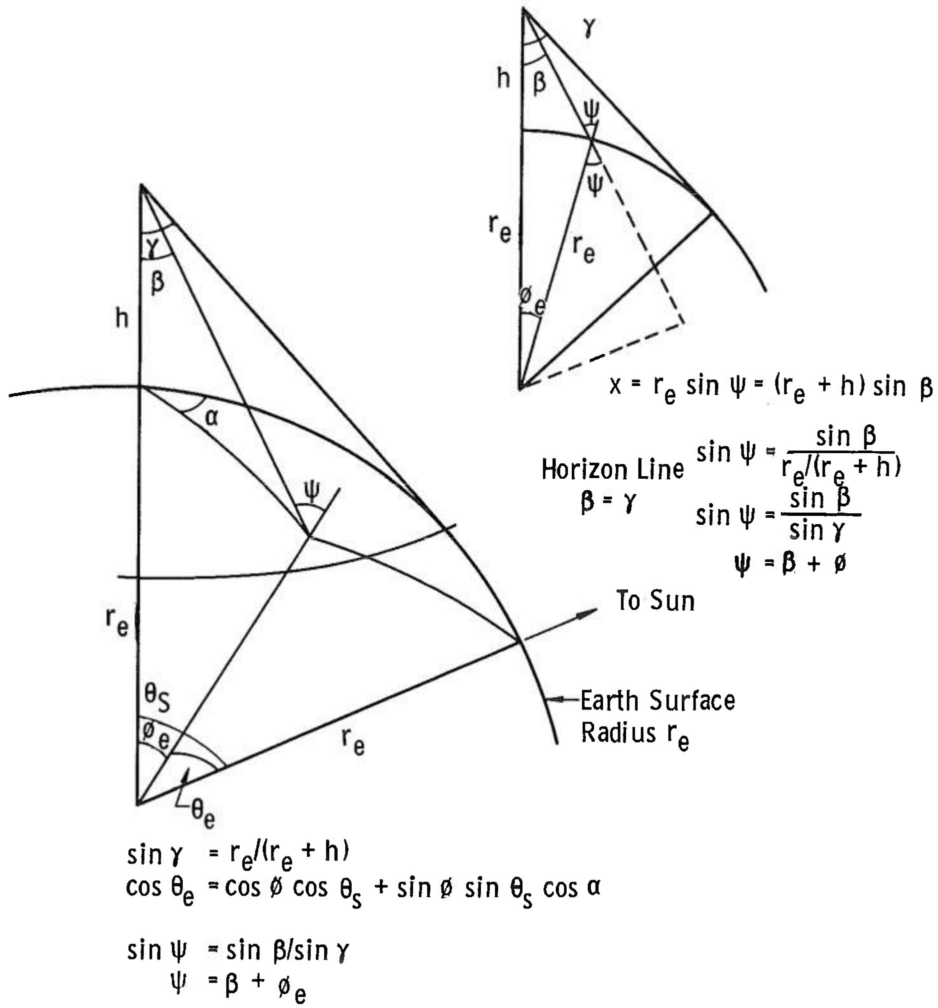


Fig. I-2 Coordinate Relations for Albedo and Earth Radiance Calculation

At the element ΔA_2. ψ is the angle between the normal to A_1 and the direction of ΔA_2 as before.

At the element ΔA_2, let β be the angle between the direction to earth center and ΔA_1. Angle β is a "nadir" angle.

Then the following relations hold (Fig. I-2):

$$\psi = \beta + \phi$$

$$\sin \psi = \sin \beta / \sin \gamma$$

$$\cos \theta_e = \cos \theta_s \cos \phi + \sin \theta_s \sin \phi_e \cos \alpha$$

where α is the angle about the line from ΔA_2 to earth center measured from the plane including this line and the sun. Angle α is the azimuth angle (Fig. I-3).

Now the element ΔA_1 is described in spherical coordinates having polar axis along the earth-to- ΔA_2 line, longitude α, and co-latitude ϕ_e:

$$\Delta A_1 = r_e^2 \sin \phi_e \, d\alpha \, d\phi_e$$

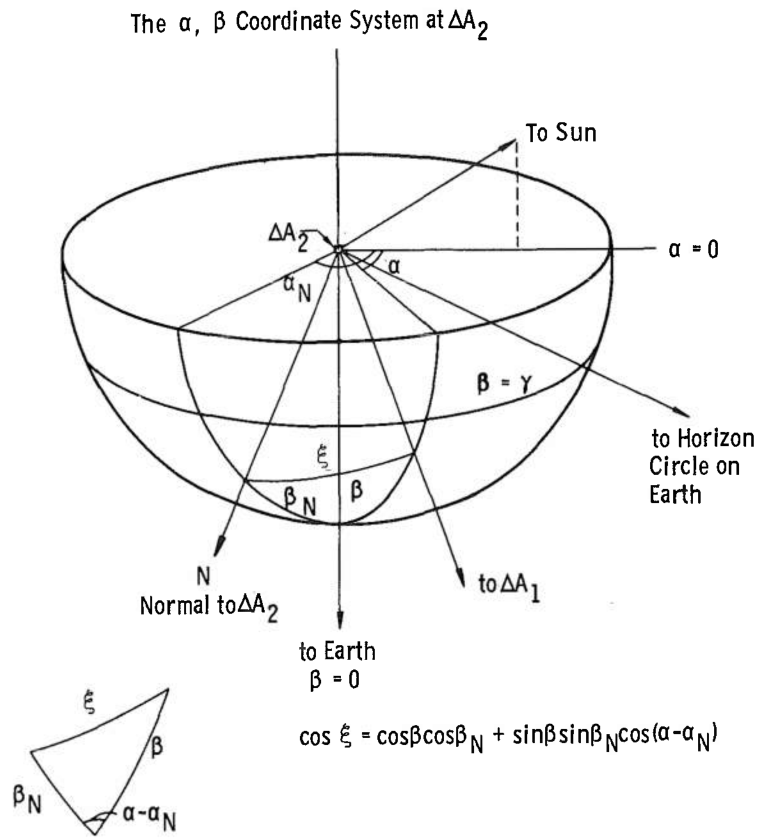


Fig. I-3 Albedo-Radiance Integration Coordinates

Similarly, we may describe a spherical area element in terms of α, β, and ρ_e from ΔA_2:

$$\Delta A_\rho = \rho_e^2 \sin\beta \, d\beta \, d\alpha$$

Hence any point may be described by either (r_e, ϕ_e, α) or (ρ_e, β, α) at ΔA_1, and we find that ΔA_ρ and ΔA_1 are related by simple projective properties:

$$\Delta A_1 \cos\psi = \Delta A_\rho$$

so that

$$\Delta A_1 = \frac{\rho_e^2 \sin\beta \, d\beta \, d\alpha}{\cos\psi}$$

We may now write the intensity at ΔA_2 caused by reflection from ΔA_1 :

$$\frac{\Delta I_2}{\Delta A_2} = \frac{A_e \; I_s}{\pi} \cos\theta_e \; \cos\psi \; \frac{\cos\xi}{\rho_e^{\,2}} \; \frac{\rho_e^{\,2} \sin\beta}{\cos\psi} \; d\alpha \; d\beta$$

$$= \frac{A_e \; I_s}{\pi} \cos\theta_e \; \cos\xi \; \sin\beta \; d\beta \; d\alpha$$

Integrating over the interior of the horizon circle, we obtain

$$\frac{I_2}{\Delta A_2} (\Delta A_2, \theta_s, \gamma, \xi) = \frac{A_e \; I_s}{\pi} \int_{\beta=0}^{\gamma} \int_{\alpha_{min}}^{\alpha_{max}} \cos\theta_e \; (d, \beta, \theta_s, \gamma) \; \cos\xi \; \sin\beta \; d\beta \; d\alpha$$

where α_{min}, α_{max} may be functions of β, and there may be several distinct regions or functions.

In the spherical coordinates α, β, any orientation of surface element may be described by the direction angles of its normal, α_N, β_N. Then the angle ξ is found from

$$\cos\xi = \cos\beta \; \cos\beta_N + \sin\beta \; \sin\beta_N \; \cos(\alpha - \alpha_N)$$

Since

$$\theta_e = \psi - \beta, \quad \text{and} \quad \sin\psi = \sin\beta / \sin\gamma$$

$$\cos\theta_e = \sin\psi \, (\cos\theta_s \; \sin\beta + \sin\theta_s \; \cos\beta \; \cos\alpha)$$

$$+ \cos\psi \, (\cos\theta_s \; \cos\beta - \sin\theta_s \; \sin\beta \; \cos\alpha)$$

The intensity integrand is completely expressible in the two variables α, β, and the configuration parameters θ_s, γ, α_N, β_N.

At this stage it is possible to integrate numerically by letting α range from 0 to 2π and β range from 0 to γ, provided that

$$\cos\xi \geq 0$$

$$\cos\theta_e \geq 0$$

$$\cos\psi \geq 0$$

and using $(= 0)$ for any (α, β) violating these conditions.

Ignoring for the moment the constant $A_e I_s/\pi$, we have the following terms to be integrated over α and β :

1. $\cos\beta_N \; \cos\theta_s \qquad \cos^2\beta \; \sin\beta \; \cos\psi$
2. $-\cos\beta_N \; \sin\theta_s \qquad \cos\beta \; \sin^2\beta \; \cos\psi \; \cos\alpha$
3. $\cos\beta_N \; \cos\theta_s \qquad \cos\beta \; \sin^2\beta \; \sin\psi$
4. $\cos\beta_N \; \sin\theta_s \qquad \cos^2\beta \; \sin\beta \; \sin\psi \; \cos\alpha$

5.	$\sin\beta_N$	$\cos\theta_s$	$\cos\alpha_N$	$\sin^2\beta$	$\cos\beta$	$\cos\psi$	$\cos\alpha$	
6.	$\sin\beta_N$	$\cos\theta_s$	$\sin\alpha_N$	$\sin^2\beta$	$\cos\beta$	$\cos\psi$	$\sin\alpha$	
7.	$-\sin\beta_N$	$\sin\theta_s$	$\cos\alpha_N$	$\sin^3\beta$		$\cos\psi$	$\cos^2\alpha$	
8.	$-\sin\beta_N$	$\sin\theta_s$	$\sin\alpha_N$	$\sin^3\beta$		$\cos\psi$	$\sin\alpha$	$\cos\alpha$
9.	$\sin\beta_N$	$\cos\theta_s$	$\cos\alpha_N$	$\sin^3\beta$		$\sin\psi$	$\cos\alpha$	
10.	$\sin\beta_N$	$\cos\theta_s$	$\sin\alpha_N$	$\sin^3\beta$		$\sin\psi$	$\sin\alpha$	
11.	$\sin\beta_N$	$\sin\theta_s$	$\cos\alpha_N$	$\sin^2\beta$	$\cos\beta$	$\sin\psi$	$\cos^2\alpha$	
12.	$\sin\beta_N$	$\sin\theta_s$	$\sin\alpha_N$	$\sin^2\beta$	$\cos\beta$	$\sin\psi$	$\sin\alpha$	$\cos\alpha$

Note: $\sin\psi = \sin\beta/\sin\gamma$

As long as there are no boundaries of earth surface for which $\alpha = \alpha(\beta)$, so that α can range from 0 to 2π, we may integrate relative to α and obtain simple results. Integrals (1, 3) do not contain α so the integration results in a factor 2π. Integrals (2, 4, 5, 6, 8, 9, 10, 12) contain only $\sin\alpha$ or $\cos\alpha$ and vanish. Integrals (7, 11) contain $\cos^2\alpha$ or $\cos\alpha \quad \sin\alpha$ and result in a factor of π.

These conditions are satisfied as long as we have both

$$\beta_N \leq \frac{\pi}{2} - \gamma$$

$$\theta_s \leq \gamma$$

If $\beta_N \geq \pi/2 + \gamma$ <u>or</u> $\theta_s \geq \pi - \gamma$, the earlier conditions on $\cos\xi$ or $\cos\theta_e$ are violated and the entire integral $(I_2/\Delta A_2)$ vanishes.

For
$$\frac{\pi}{2} - \gamma < \beta_N < \frac{\pi}{2} + \gamma$$

and/or
$$\gamma < \theta_s < \pi - \gamma$$

there exist boundaries of form $\alpha(\beta)$, and the integration becomes complicated. The integration is bounded by arcs of one, two, or three curves of $\alpha(\beta)$, whose intersections are generally given by implicit functions. A first integration may be done formally; expressions result for which the integrals are not available in closed form.

Numerical integration may be accomplished in an easily comprehended manner by referring to the earlier integral expression. The product $\cos\xi \cos\theta_e \sin\beta$ may be calculated term by term, and in addition, the expression $\cos\psi$ can be evaluated to ensure that the conditions

$$\left.\begin{array}{l}\cos\xi \\ \cos\theta_e \\ \cos\psi\end{array}\right\} \geq 0$$

are satisfied. For some value combinations of a_N, β_N, θ_s the integrations can be carried out.

APPENDIX II
FIRST INTEGRATION IN NADIR ANGLE β

Table II-I displays the twelve possible integrands, with their parameter coefficients. The last eight of these may be grouped in pairs and combined by use of the identity

$$\cos \alpha \cos \alpha_N + \sin \alpha \sin \alpha_N = \cos (\alpha - \alpha_N)$$

Further grouping can then be performed based on the formal similarity of integrands. We use the numbering of Table II-I to identify integrands and the following definitions of parameter functions.

$$A_1 = \sin \beta_N \sin \theta_s \cos \alpha \cos (\alpha - \alpha_N)$$

$$A_2 = \cos \beta_N \cos \theta_s$$

$$A_3 = \sin \beta_N \cos \theta_s \cos (\alpha - \alpha_N)$$

$$A_4 = \cos \beta_N \sin \theta_s \cos \alpha$$

$$(1) \quad A_2 \int \cos^2 \beta \sin \beta \, (1 - \sin^2 \beta / \sin^2 \gamma)^{1/2} \, d\beta$$

$$(2) \quad -A_4 \int \cos \beta \sin^2 \beta \, (1 - \sin^2 \beta / \sin^2 \gamma)^{1/2} \, d\beta$$

$$(3) \quad A_2 \int \cos \beta \, (\sin^3 \beta / \sin \gamma) \, d\beta$$

$$(4) \quad A_4 \int \cos^2 \beta \, (\sin^2 \beta / \sin \gamma) \, d\beta$$

$$(5, 6) \quad A_3 \int \cos \beta \sin^2 \beta \, (1 - \sin^2 \beta / \sin^2 \gamma)^{1/2} \, d\beta$$

$$(7, 8) \quad -A_1 \int \sin^3 \beta \, (1 - \sin^2 \beta / \sin^2 \gamma)^{1/2} \, d\beta$$

$$(9, 10) \quad A_3 \int (\sin^4 \beta / \sin \gamma) \, d\beta$$

$$(11, 12) \quad A_1 \int \cos \beta \, (\sin^3 \beta / \sin \gamma) \, d\beta$$

TABLE II-1
INTEGRAND FORMS FOR ALBEDO

1.	$\cos\beta_N$	$\cos\theta_s$		$\cos^2\beta$	$\sin\beta$	$\cos\psi$	
2.	$-\cos\beta_N$	$\sin\theta_s$		$\cos\beta$	$\sin^2\beta$	$\cos\psi$	$\cos\alpha$
3.	$\cos\beta_N$	$\cos\theta_s$		$\cos\beta$	$\sin^2\beta$	$\sin\psi$	
4.	$\cos\beta_N$	$\sin\theta_s$		$\cos^2\beta$	$\sin\beta$	$\sin\psi$	$\cos\alpha$
5.	$\sin\beta_N$	$\cos\theta_s$	$\cos\alpha_N$	$\sin^2\beta$	$\cos\beta$	$\cos\psi$	$\cos\alpha$
6.	$\sin\beta_N$	$\cos\theta_s$	$\sin\alpha_N$	$\sin^2\beta$	$\cos\beta$	$\cos\psi$	$\sin\alpha$
7.	$-\sin\beta_N$	$\sin\theta_s$	$\cos\alpha_N$	$\sin^3\beta$		$\cos\psi$	$\cos^2\alpha$
8.	$-\sin\beta_N$	$\sin\theta_s$	$\sin\alpha_N$	$\sin^3\beta$		$\cos\psi$	$\sin\alpha\cos\alpha$
9.	$\sin\beta_N$	$\cos\theta_s$	$\cos\alpha_N$	$\sin^3\beta$		$\sin\psi$	$\cos\alpha$
10.	$\sin\beta_N$	$\cos\theta_s$	$\sin\alpha_N$	$\sin^3\beta$		$\sin\psi$	$\sin\alpha$
11.	$\sin\beta_N$	$\sin\theta_s$	$\cos\alpha_N$	$\sin^2\beta$	$\cos\beta$	$\sin\psi$	$\cos^2\alpha$
12.	$\sin\beta_N$	$\sin\theta_s$	$\sin\alpha_N$	$\sin^2\beta$	$\cos\beta$	$\sin\psi$	$\sin\alpha\cos\alpha$

Note: $\sin\psi = \sin\beta / \sin\gamma$

Regrouping for formal similarity

$$(1, 7, 8) \qquad (A_2 + A_1) \int \cos^2\beta\,(1 - \sin^2\beta/\sin^2\gamma)^{1/2} \sin\beta\, d\beta$$
$$- A_1 \int (1 - \sin^2\beta/\sin^2\gamma)^{1/2} \sin\beta\, d\beta$$

$$(2, 5, 6) \qquad (A_3 - A_4) \int \sin^2\beta\,(1 - \sin^2\beta/\sin^2\gamma)^{1/2} \cos\beta\, d\beta$$

$$(3, 11, 12) \qquad (A_1 + A_2) \int (\sin^3\beta/\sin\gamma) \cos\beta\, d\beta$$

$$(4, 9, 10) \qquad A_4 \int (\sin^2\beta/\sin\gamma)\, d\beta + (A_3 - A_4) \int (\sin^4\beta/\sin\gamma)\, d\beta$$

We proceed to integrate, first making the following substitutions

(1, 7, 8) Let

$$\sin\gamma = G \qquad \cos\gamma = B \qquad \cos\beta = x$$

and note that

$$\sin^2\gamma - \sin^2\beta = \cos^2\beta - \cos^2\gamma$$

$$\sin\beta\, d\beta = -d\cos\beta = -dx$$

Then we obtain

$$-\frac{(A_2 + A_1)}{G} \int x^2 (x^2 - B^2)^{1/2} dx + \frac{A_1}{G} \int (x^2 - B^2)^{1/2} dx$$

$$= \frac{(A_2 + A_1)}{G} \left[\frac{x}{4}(x^2 - B^2)^{3/2} + \frac{B^2}{8} x (x^2 - B^2)^{1/2} - \frac{B^4}{8} \ln\,(x + (x^2 - B^2)^{1/2})\right]$$
$$+ \frac{A_1}{G} \left[\frac{x}{2}(x^2 - B^2)^{1/2} - \frac{B^2}{2} \ln\,(x + (x^2 - B^2)^{1/2})\right]$$

$$= -\frac{(A_2 + A_1)}{G} \frac{x}{4} (x^2 - B^2)^{3/2}$$
$$+ \left[\frac{A_1}{2G} - \frac{A_1 + A_2}{G} \frac{B^2}{8}\right] \left\{x (x^2 - B^2)^{1/2} - B^2 \ln\left(x + (x^2 - B^2)^{1/2}\right)\right\}$$

Finally

$$(1, 7, 8) \quad \frac{1}{4G}\left\{\left[2A_1 - \frac{1}{2}(A_1 + A_2) B^2\right]\left[x (x^2 - B^2)^{1/2} - B^2 \ln\left(x + (x^2 - B^2)^{1/2}\right)\right]\right.$$
$$\left. - (A_1 + A_2)\, x (x^2 - B^2)^{3/2}\right\}$$

(2, 5, 6) Let

$$\sin\gamma = G \qquad \sin\beta = y \qquad \cos\psi = z$$

where

$$\sin\psi = \sin\beta/\sin\gamma$$

and note

$$\cos\beta \; d\beta = d \; \sin\beta = dy$$

Then we obtain

$$\frac{(A_3 - A_4)}{G} \int y^2 (G^2 - y^2)^{1/2} \, dy$$

$$= \frac{(A_3 - A_4)}{G} \left[-\frac{y}{4} (G^2 - y^2)^{3/2} + \frac{G^2}{8} \left\{ y (G^2 - y^2)^{1/2} + G^2 \sin^{-1} \frac{y}{G} \right\} \right]$$

$$= \frac{A_4 - A_3}{4G} \left[y (G^2 - y^2)^{3/2} - \frac{G^2}{2} \left\{ y (G^2 - y^2)^{1/2} + G^2 \psi \right\} \right]$$

$$= \frac{A_4 - A_3}{4G} \left[y z^3 - \frac{1}{2} (y z + G\psi) \right] G^3$$

(3, 11, 12) Let

$$G = \sin\gamma$$

then

$$(A_2 + A_1) \int \frac{\sin^3\beta \, \cos\beta \, d\beta}{\sin\gamma} = \frac{(A_2 + A_1)}{4G} \sin^4\beta$$

(4, 9, 10) Let

$$G = \sin\gamma$$

then

$$\frac{A_4}{G} \int \sin^2\beta \, d\beta + \frac{(A_3 - A_4)}{G} \int \sin^4\beta \, d\beta$$

$$= \frac{A_4}{G} \left[\frac{1}{2} (\beta - \sin\beta \, \cos\beta) \right]$$

$$+ \frac{(A_3 - A_4)}{G} \left[-\frac{\sin^3\beta \, \cos\beta}{4} + \frac{3}{4} \left\{ \frac{1}{2} (\beta - \sin\beta \, \cos\beta) \right\} \right]$$

$$= \frac{1}{4G} \Big[(A_4 - A_3) \sin^3\beta \, \cos\beta$$

$$+ \frac{1}{2} (A_4 + 3A_3) (\beta - \sin\beta \, \cos\beta) \Big]$$

Now all groups have a common factor $1/(4G)$; this factor is ignored in practice until final calculation of the albedo view factor, which is the calculated value of the integral multiplied by

$$1/(4\pi \sin\gamma)$$

The actual albedo illumination intensity is then gotten by multiplying by the albedo A_e and solar constant I_s.

APPENDIX III
INTERIOR INTERSECTIONS OF BOUNDARY CURVES

Intercepts of C_θ with C_ξ are defined by the system of equations

$$\cos \alpha = -1/(\tan \theta_s \tan \phi_e) \tag{III-1}$$

$$\cos(\alpha - \alpha_N) = -1/(\tan \beta_N \tan \beta) \tag{III-2}$$

with the conditions

$$\phi_e = \psi - \beta \tag{III-3}$$

$$\sin \psi = \sin \beta / \sin \gamma \tag{III-4}$$

thus all four parameters θ_s, γ, α_N, and β_N are involved. From the conditions of Eqs. (III-3) and (III-4) we derive the relation

$$\tan \beta = \frac{\sin \phi_e \sin \gamma}{1 - \sin \gamma \cos \phi_e} \tag{III-5}$$

We expand the left side of Eq. (III-2) and use Eq. (III-5) on the right of Eq. (III-2) to obtain

$$C \cos \alpha + S \sin \alpha = (G \cos \phi_e - 1)/BG \sin \phi_e \tag{III-6}$$

where

$$B = \tan \beta_N$$

$$C = \cos \alpha_N, \quad G = \sin \gamma, \quad S = \sin \alpha_N$$

Let

$$T = \tan \theta_s$$

and substitute Eq. (III-1) on the left of Eq. (III-6) to obtain

$$-\frac{C \cos \phi_e}{T \sin \phi_e} + S\left(1 - \frac{\cos^2 \phi_e}{T^2 \sin^2 \phi_e}\right)^{1/2} = \frac{G \cos \phi_e - 1}{BG \sin \phi_e}$$

Now write the middle term as

$$\frac{S}{T \sin \phi_e}(T^2 \sin^2 \phi_e - \cos^2 \phi_e)^{1/2} = \frac{S}{T \sin \phi_e}\left[T^2 - (T^2 + 1)\cos^2 \phi_e\right]^{1/2}$$

and factor out $\sin \phi_e$ assuming $\phi_e \neq 0$. Isolate the radical on the left side and obtain, by squaring and rearranging,

$$G^2\left[(B^2+1)(T^2+1) - (1 - BCT)^2\right] \cos \phi_e - 2T(BC + T)\, G \cos \phi_e + T^2(1 - S^2B^2G^2) = 0 \tag{III-7}$$

Solving this quadratic for $\cos\phi_e$, we find

$$\cos\phi_e = \frac{T}{G} \cdot \frac{(BC+T) \pm \left\{ (BC+T)^2 - \left[(B^2+1)(T^2+1) - (1-BCT)^2 \right] (1 - S^2B^2G^2) \right\}^{1/2}}{(B^2+1)(T^2+1) - (1-BCT)^2}$$

(III-8)

From these two (±) values of $\cos\phi_e$ we may find corresponding values for $\sin\phi_e$, noting that $\phi_e \leq \frac{\pi}{2} - \gamma$ always, by definition. Use the values of $\cos\phi_e$, $\sin\phi_e$ in Eq. (III-5) to obtain two corresponding values of $\tan\beta$, hence β, noting that for this purpose $\beta \leq \gamma$. At the same time, Eq. (III-1) allows evaluation of the two values of α, and the intersection points for $C\theta$ and $C\xi$ can be found. These points are labeled (α_5, β_5) and (α_6, β_6).

Now the discriminant of Eq. (III-8) provides indications of the presence of two, one, or no intersections of $C\theta$ and $C\xi$.

From the fact that (α_5, α_6) contains α values of intersection for two curves defined between both pairs (α_1, α_2) and (α_3, α_4), of necessity (α_5, α_6) values lie inside both ranges (α_1, α_2) and (α_3, α_4). That is, (α_5, α_6) values are within the α ranges of definition which are common to both $C\xi$ and $C\theta$. This is the region where, in practice, we may compute both $\beta(C\theta)$ and $\beta(C\xi)$ to select which shall be used. In this way we avoid calculation of (α_5, α_6) and avoid further complication of the selection logic.

APPENDIX IV
TRANSFORMATIONS FOR β, ϕ_e, AND ψ

Transformations for ψ, ϕ_e, and β are based on the relations

$$\psi = \phi_e + \beta$$

$$\sin\psi = \sin\beta / \sin\gamma$$

$$F = (1 + \sin^2\gamma - 2\sin\gamma\cos\phi_e)^{1/2}$$

The transformations are

$$\sin\psi = \sin\beta/\sin\gamma = \sin\phi_e/F$$

$$\cos\psi = (\sin^2\gamma - \sin^2\beta)^{1/2}/\sin\gamma$$

$$= (\cos\phi_e - \sin\gamma)/F$$

$$\sin\beta = \sin\gamma\ \sin\phi/F = \sin\gamma\ \sin\psi$$

$$\cos\beta = (1 - \sin\gamma\ \cos\phi)/F = (1 - \sin^2\gamma\ \sin^2\psi)^{1/2}$$

$$\sin\phi_e = \left[\cos\beta - (\sin^2\gamma - \sin^2\beta)^{1/2}\right]\sin\beta/\sin\gamma$$

$$= \sin\psi\left[(1 - \sin^2\gamma\ \sin^2\psi)^{1/2} - \sin\gamma\ \cos\psi\right]$$

$$\cos\phi_e = \left[\sin^2\beta + \cos\beta\ (\sin^2\gamma - \sin^2\beta)^{1/2}\right]/\sin\gamma$$

$$= \cos\psi\ (1 - \sin^2\gamma\ \sin^2\psi)^{1/2} + \sin\gamma\ \sin^2\psi$$

$$\tan\psi = \sin\beta/(\sin^2\gamma - \sin^2\beta)^{1/2} = \sin\phi_e/(\cos\phi_e - \sin\gamma)$$

$$\tan\beta = \sin\gamma\ \sin\phi/(1 - \sin\gamma\ \cos\phi_e) = \frac{\sin\gamma\ \sin\psi}{(1 - \sin^2\gamma\ \sin^2\psi)^{1/2}}$$

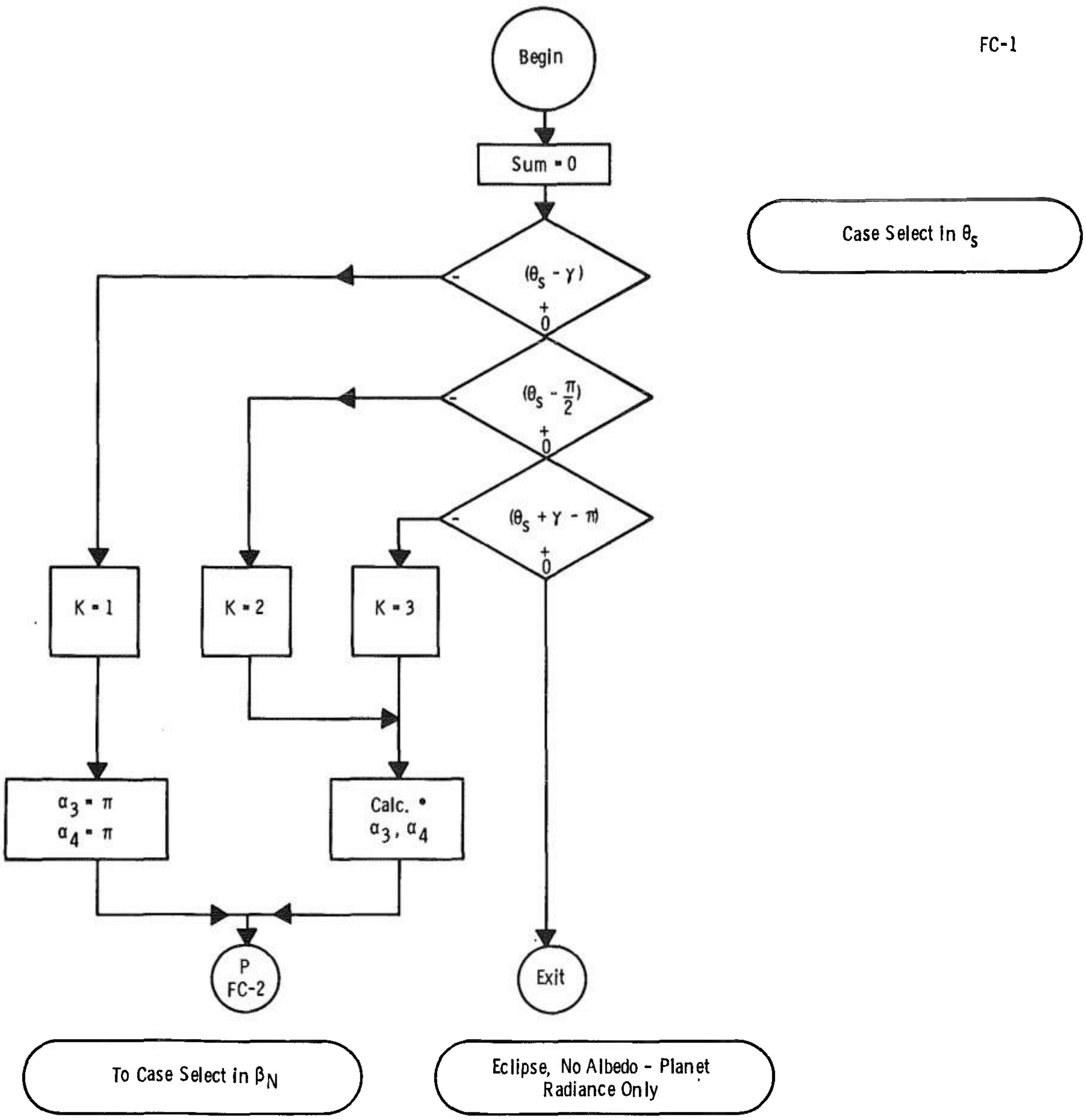
$$\tan\phi_e = \frac{[\cos\beta - (\sin^2\gamma - \sin^2\beta)^{1/2}]^{\sin\beta}}{[\sin^2\beta + \cos\beta\ (\sin^2\gamma - \sin^2\beta)^{1/2}]}$$

$$= \sin\psi\ \frac{[(1 - \sin^2\gamma\ \sin^2\psi)^{1/2} - \sin\gamma\ \cos\psi]}{[\cos\psi\ (1 - \sin^2\gamma\ \sin^2\psi)^{1/2} + \sin\gamma\ \sin^2\psi]}$$

APPENDIX V
FLOW CHART
FOR
SUBROUTINE ALBEDO (ALBDO)

APPENDIX V
FLOW CHART FOR COMPUTER PROGRAM

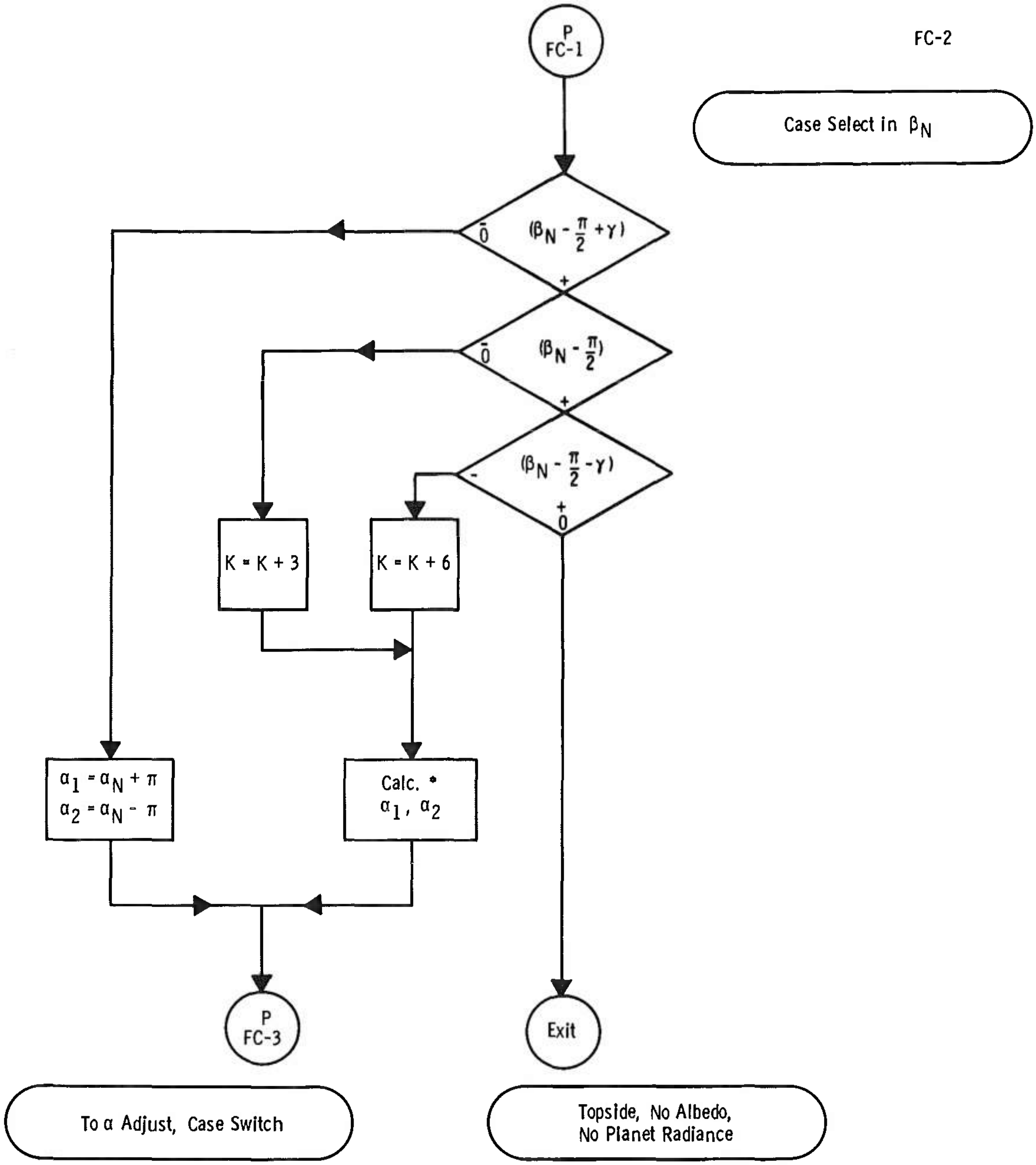
FC-1



* $\alpha_3 = \frac{\pi}{2} + \sin^{-1}\left(\frac{\tan \gamma}{\tan \theta_s}\right)$

$\alpha_4 = 2\pi - \alpha_3$

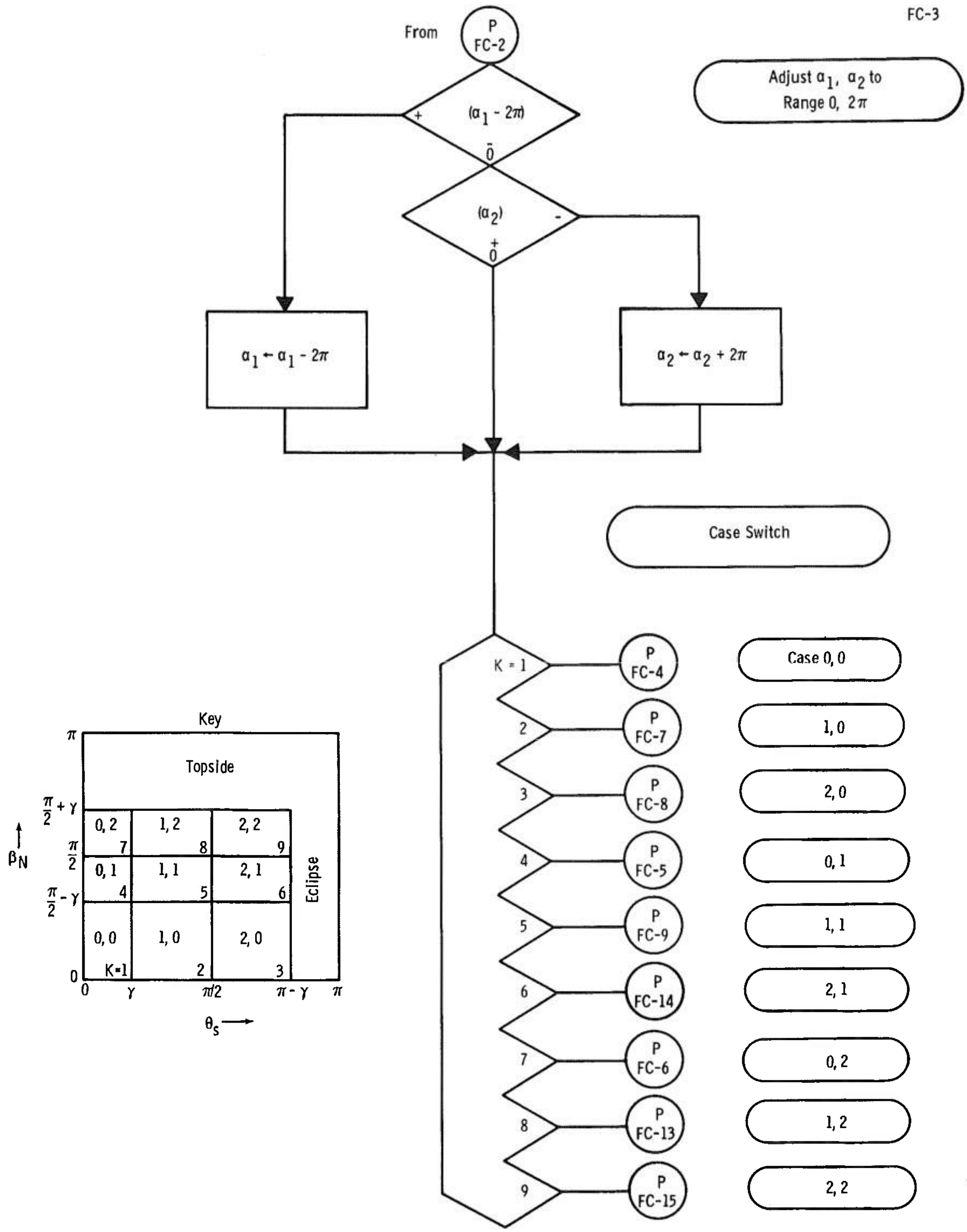
Subroutine "ALBDO" Arguments:
α_N, β_N, θ_s, γ; $\delta\alpha$

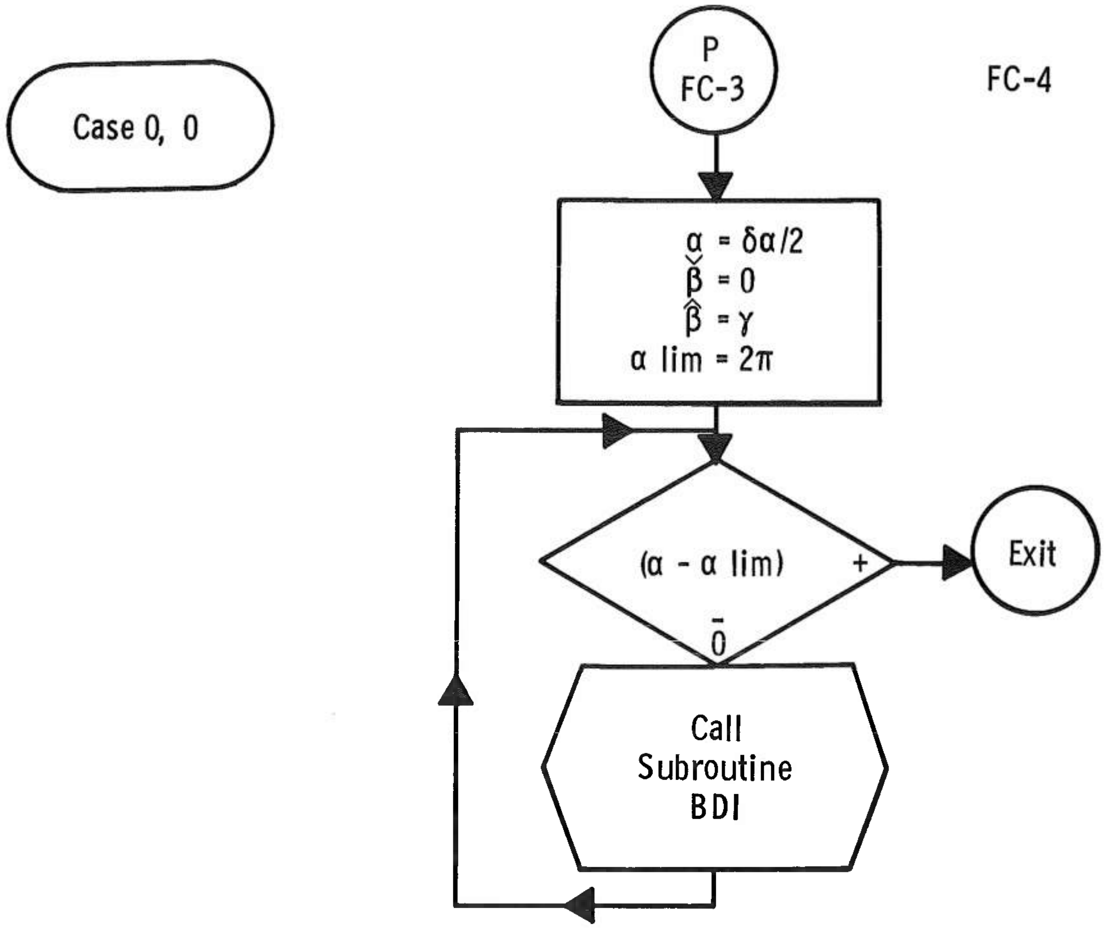


$$*\Delta\alpha = \frac{\pi}{2} + \sin^{-1}\left(\frac{1}{\tan\beta_N \tan\gamma}\right)$$

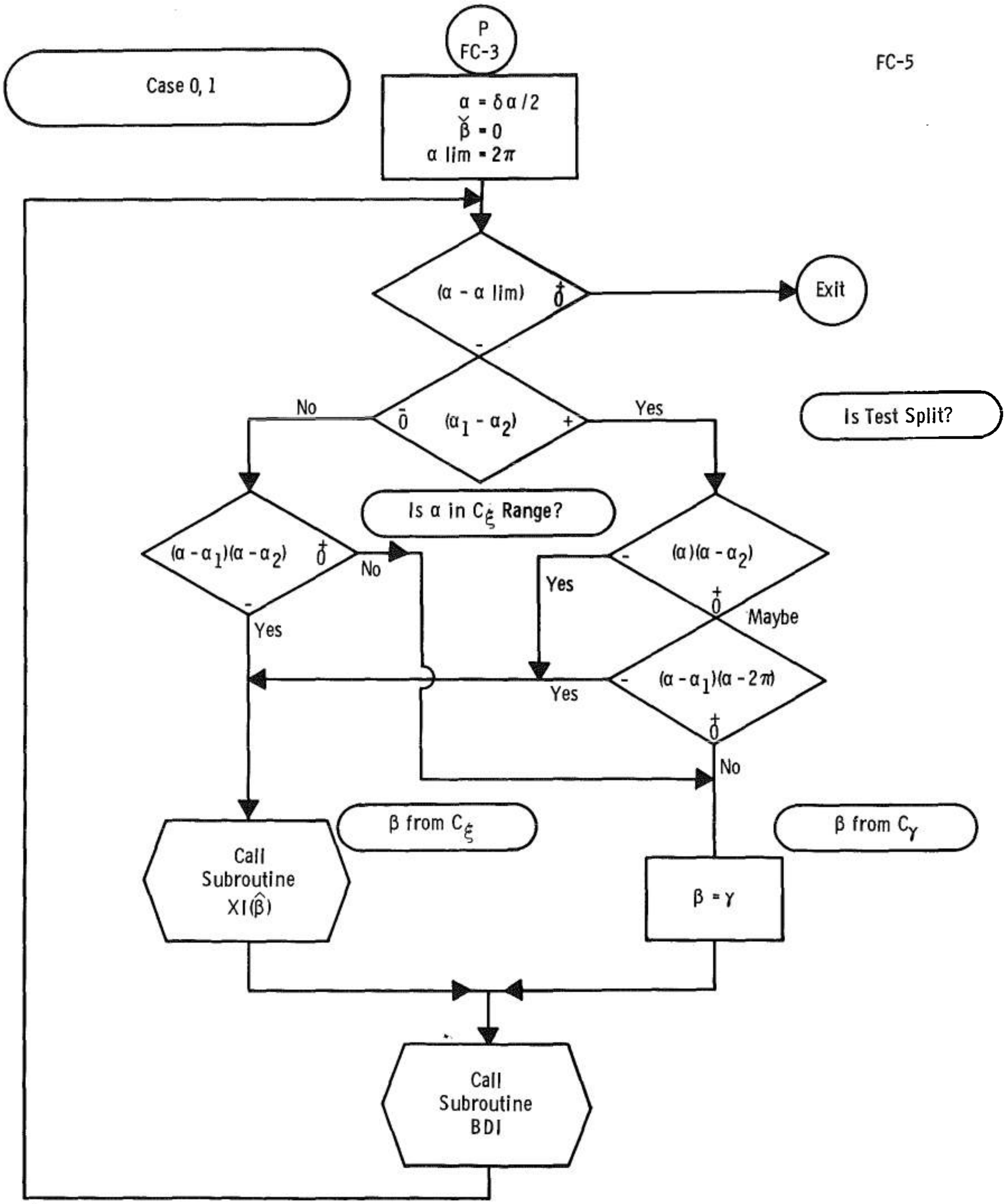
$$\alpha_1 = \alpha_N + \Delta\alpha$$

$$\alpha_2 = \alpha_N - \Delta\alpha$$


FC-3
From
P FC-2
Adjust α_1, α_2 to Range 0, 2π
$(\alpha_1 - 2\pi)$
+
$\bar{0}$
(α_2)
-
$\overset{+}{0}$
$\alpha_1 \leftarrow \alpha_1 - 2\pi$
$\alpha_2 \leftarrow \alpha_2 + 2\pi$
Case Switch
K = 1
2
3
4
5
6
7
8
9
P FC-4
P FC-7
P FC-8
P FC-5
P FC-9
P FC-14
P FC-6
P FC-13
P FC-15
Case 0, 0
1, 0
2, 0
0, 1
1, 1
2, 1
0, 2
1, 2
2, 2
Key
Topside
Eclipse
0, 2
1, 2
2, 2
0, 1
1, 1
2, 1
0, 0
1, 0
2, 0
K=1
2
3
4
5
6
7
8
9
π
$\frac{\pi}{2} + \gamma$
$\frac{\pi}{2}$
$\frac{\pi}{2} - \gamma$
0
β_N
0
γ
$\pi/2$
$\pi - \gamma$
π
θ_s



Subroutine BDI, with Arguments α, α_N, $\check{\beta}$, $\hat{\beta}$, γ, θ_s; Sum, $\delta\alpha$
Evaluates the First Integral Between $\hat{\beta}$, $\check{\beta}$
Adding Results to Sum, Advancing α by $\delta\alpha$

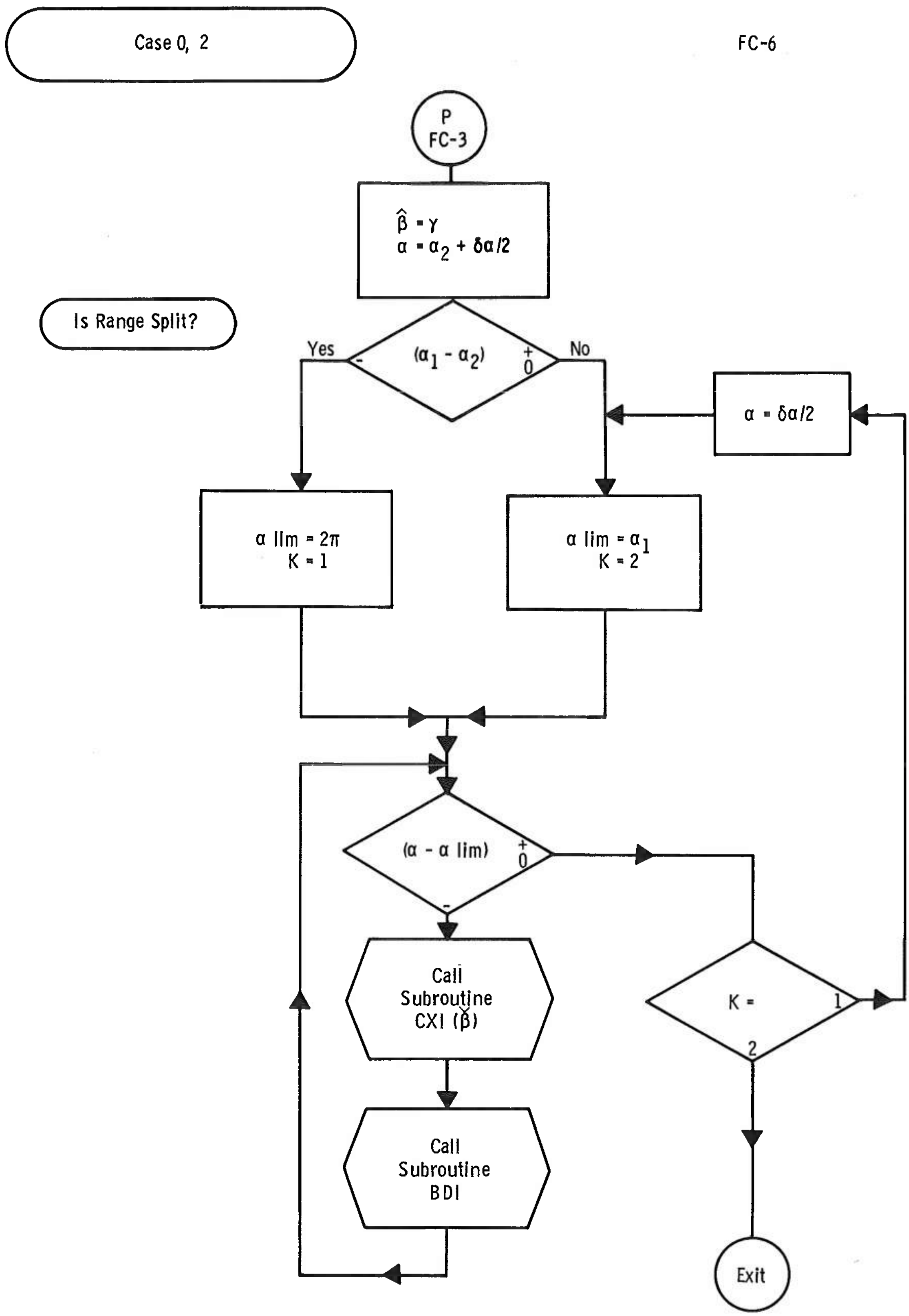


Subroutine CXI, Arguments α, α_N, β_N
Evaluates β on the Curve $C_\xi(\alpha, \beta)$
defined by

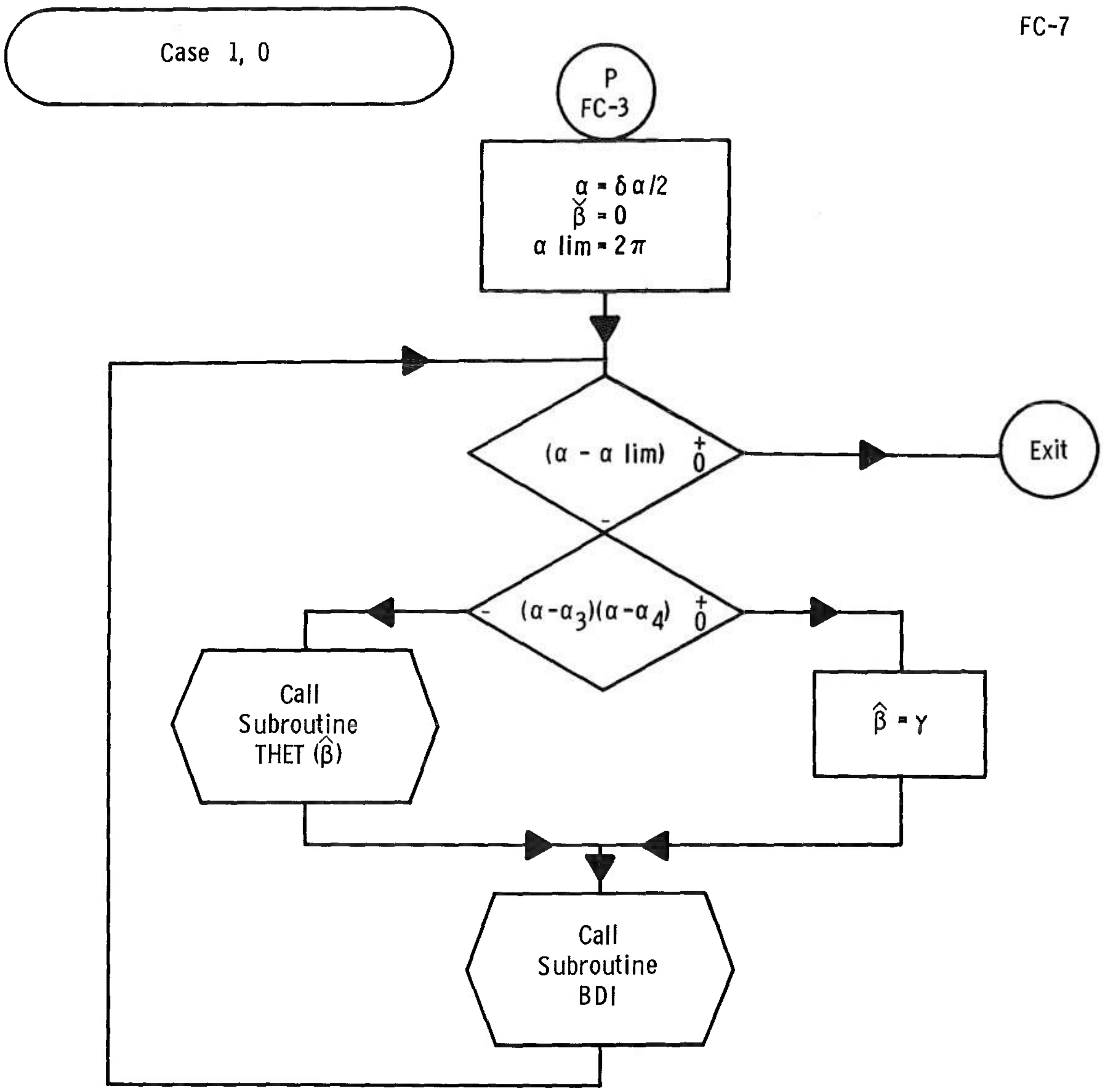
$$\cos(\alpha - \alpha_N) = -\operatorname{ctn}\beta \operatorname{ctn}\beta_N$$

or

$$\beta = \tan^{-1}\left[\frac{-1}{\tan\beta_N \cos(\alpha - \alpha_N)}\right]$$

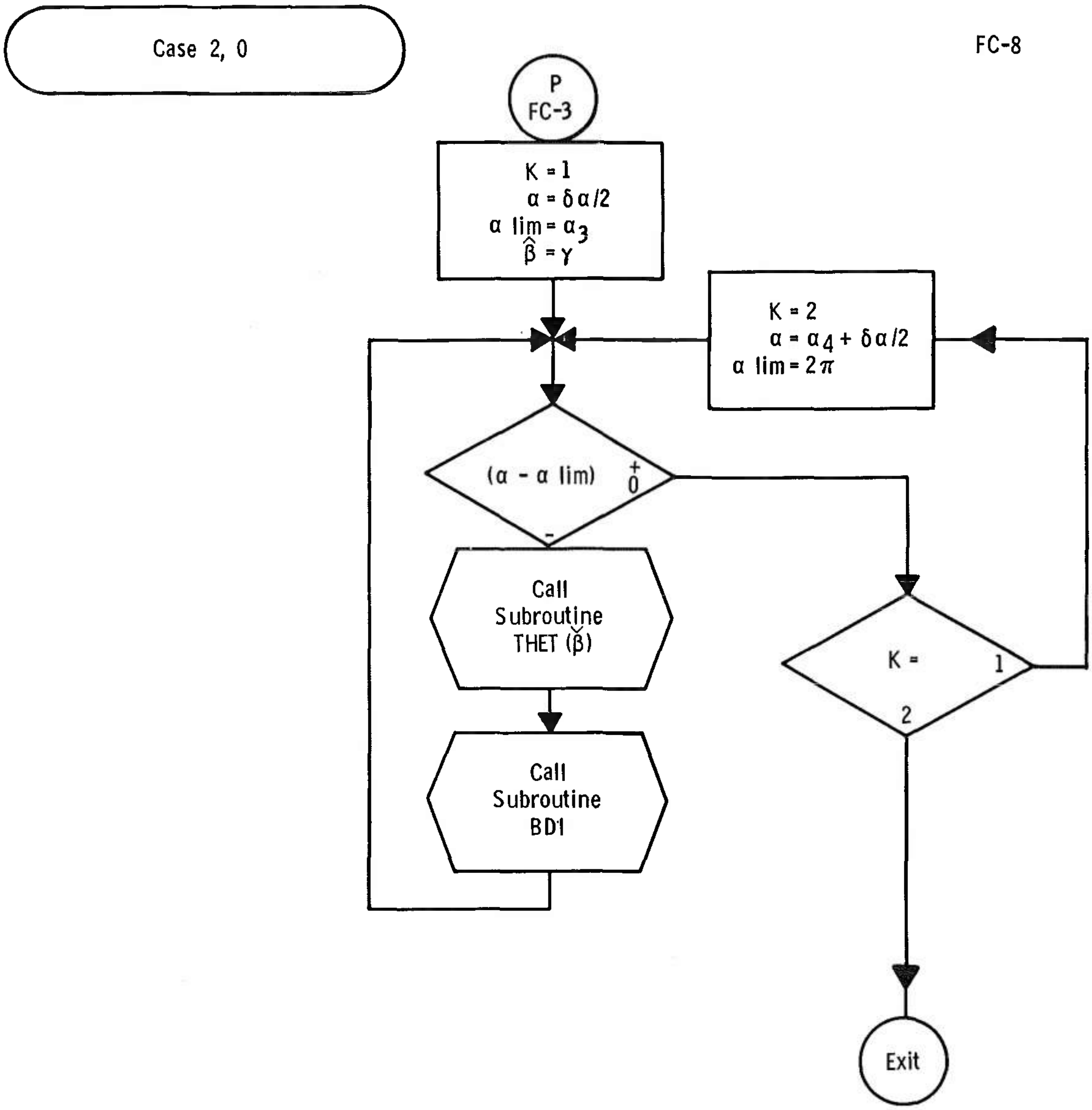

Case 0, 2
FC-6
P
FC-3
$\hat{\beta} = \gamma$
$\alpha = \alpha_2 + \delta\alpha/2$
Is Range Split?
Yes
No
$(\alpha_1 - \alpha_2)$
-
+
0
$\alpha = \delta\alpha/2$
$\alpha\ \lim = 2\pi$
K = 1
$\alpha\ \lim = \alpha_1$
K = 2
$(\alpha - \alpha\ \lim)$
+
0
-
Call
Subroutine
CXI (β)
Call
Subroutine
BDI
K =
1
2
Exit

FC-7



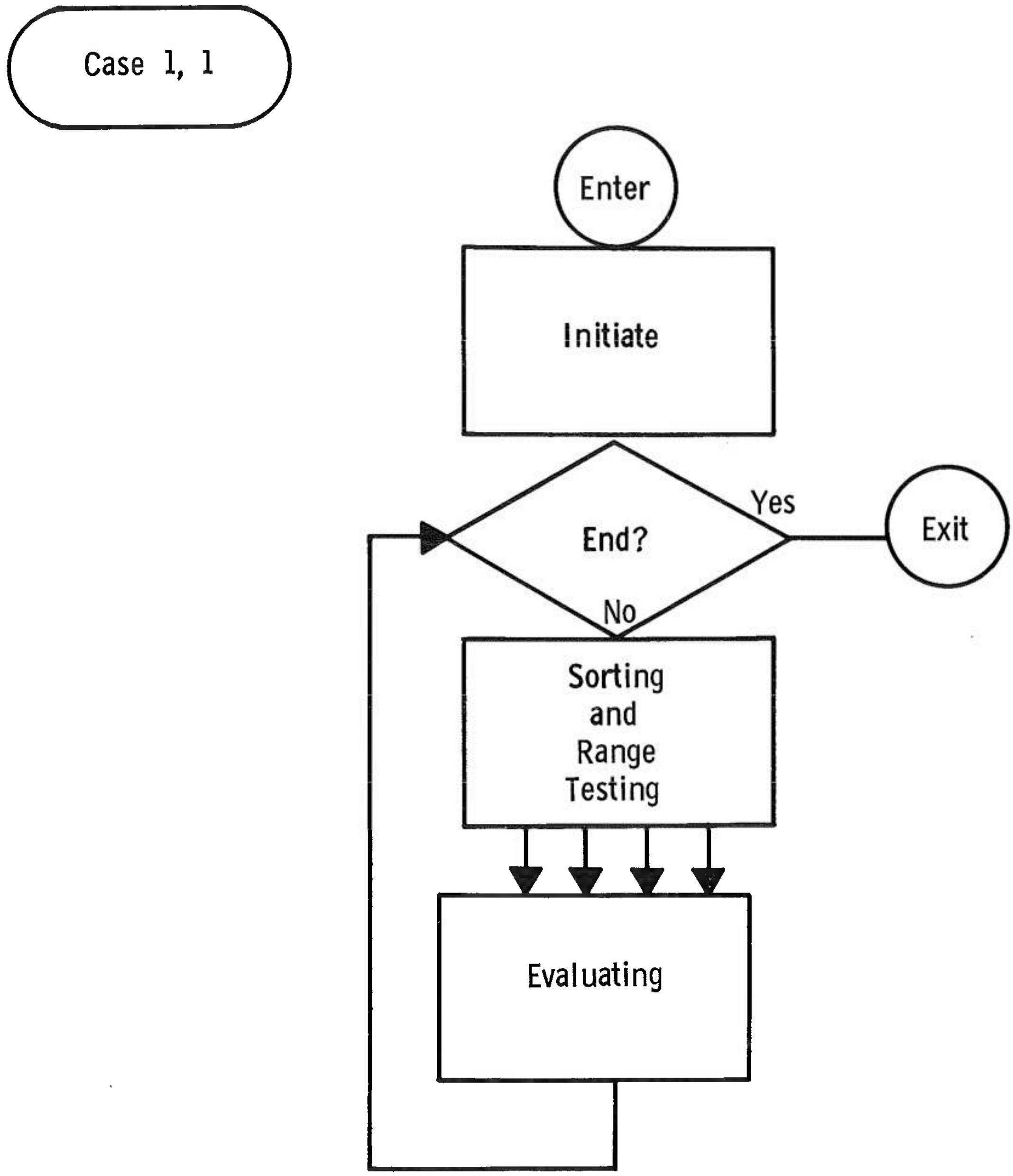
Subroutine CTH, Arguments α, θ_S, γ
Evaluates β on the Curve $C_\theta(\alpha, \beta)$
defined by the Set of Equations

$$\cos \alpha = -\operatorname{ctn} \theta_S \operatorname{ctn} \phi$$
$$\phi = \psi - \beta$$
$$\sin \psi = \sin \beta / \sin \gamma$$

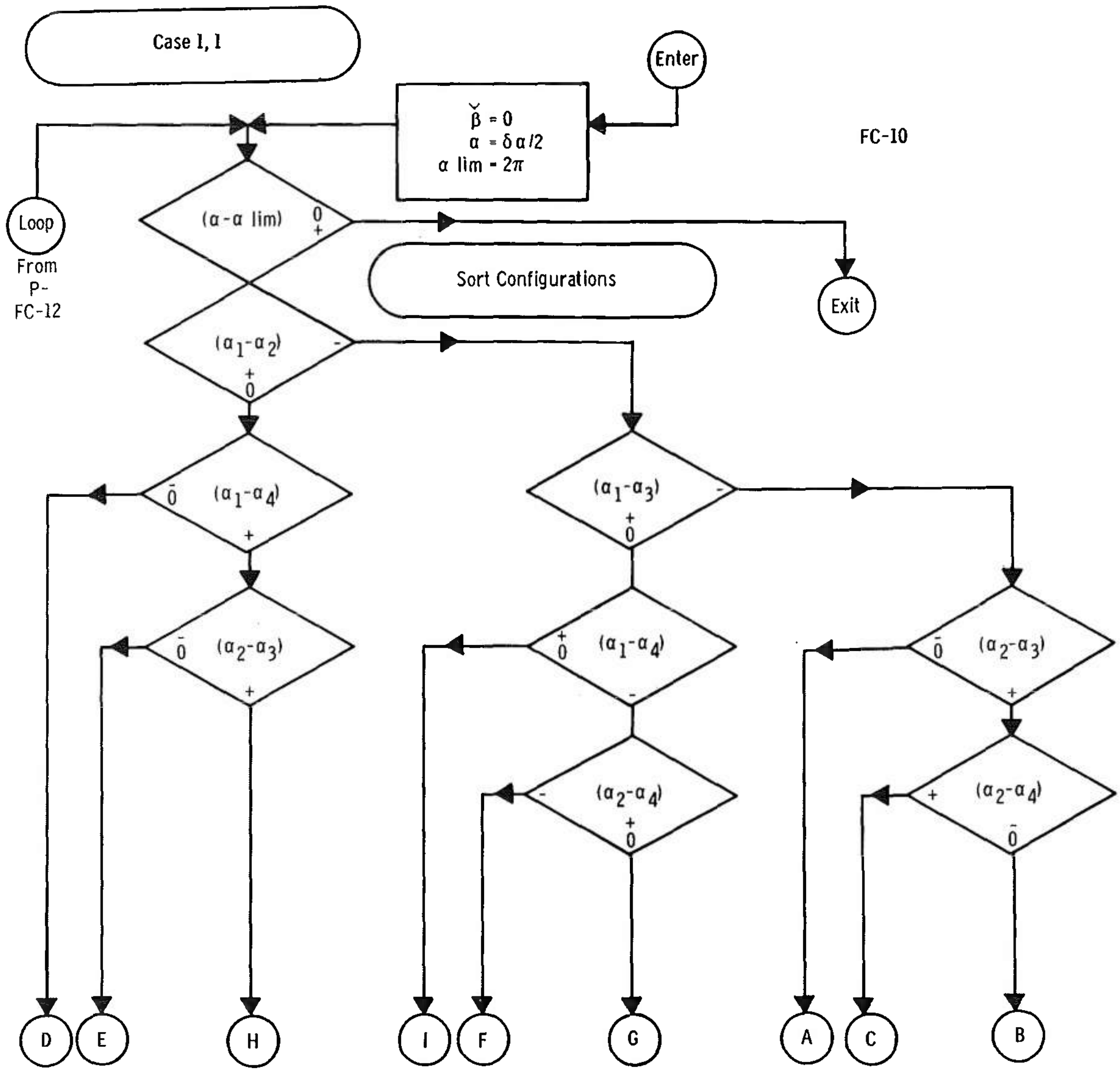


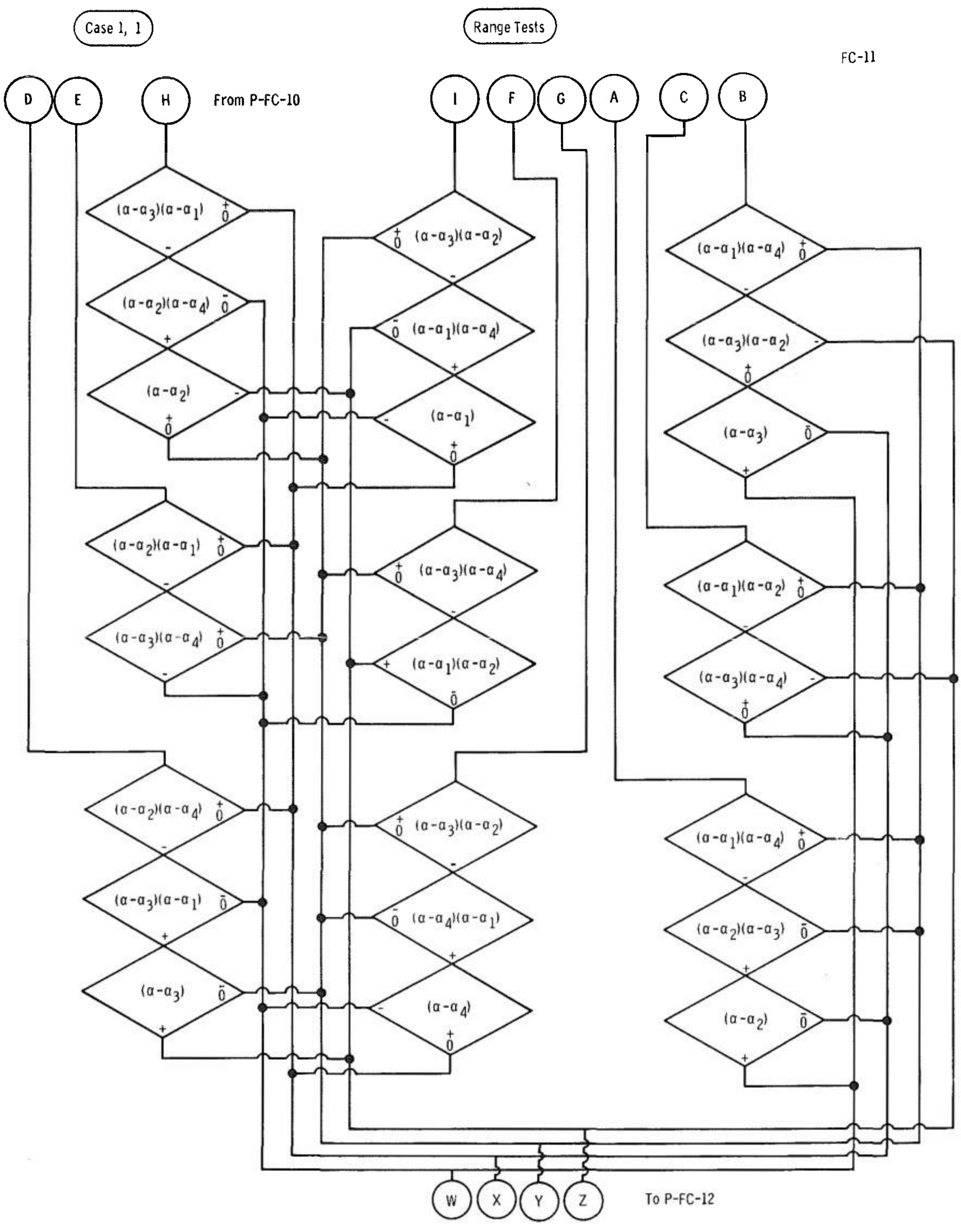
Case 2, 0
FC-8
P
FC-3
K = 1
α = δα/2
α lim = α3
β̂ = γ
K = 2
α = α4 + δα/2
α lim = 2π
(α - α lim)
+
0
-
Call
Subroutine
THET (β̆)
Call
Subroutine
BD1
K =
1
2
Exit

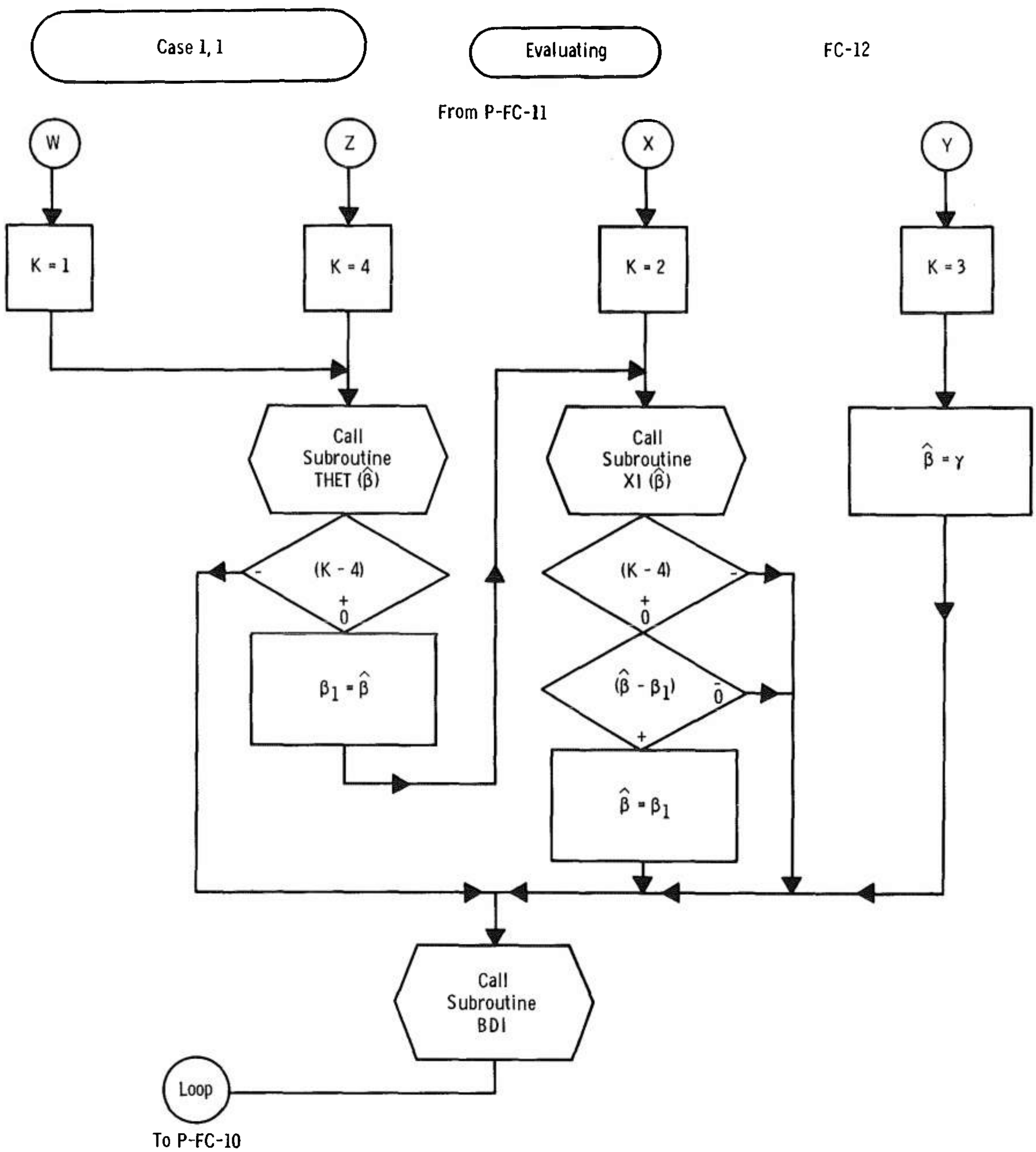
FC-9

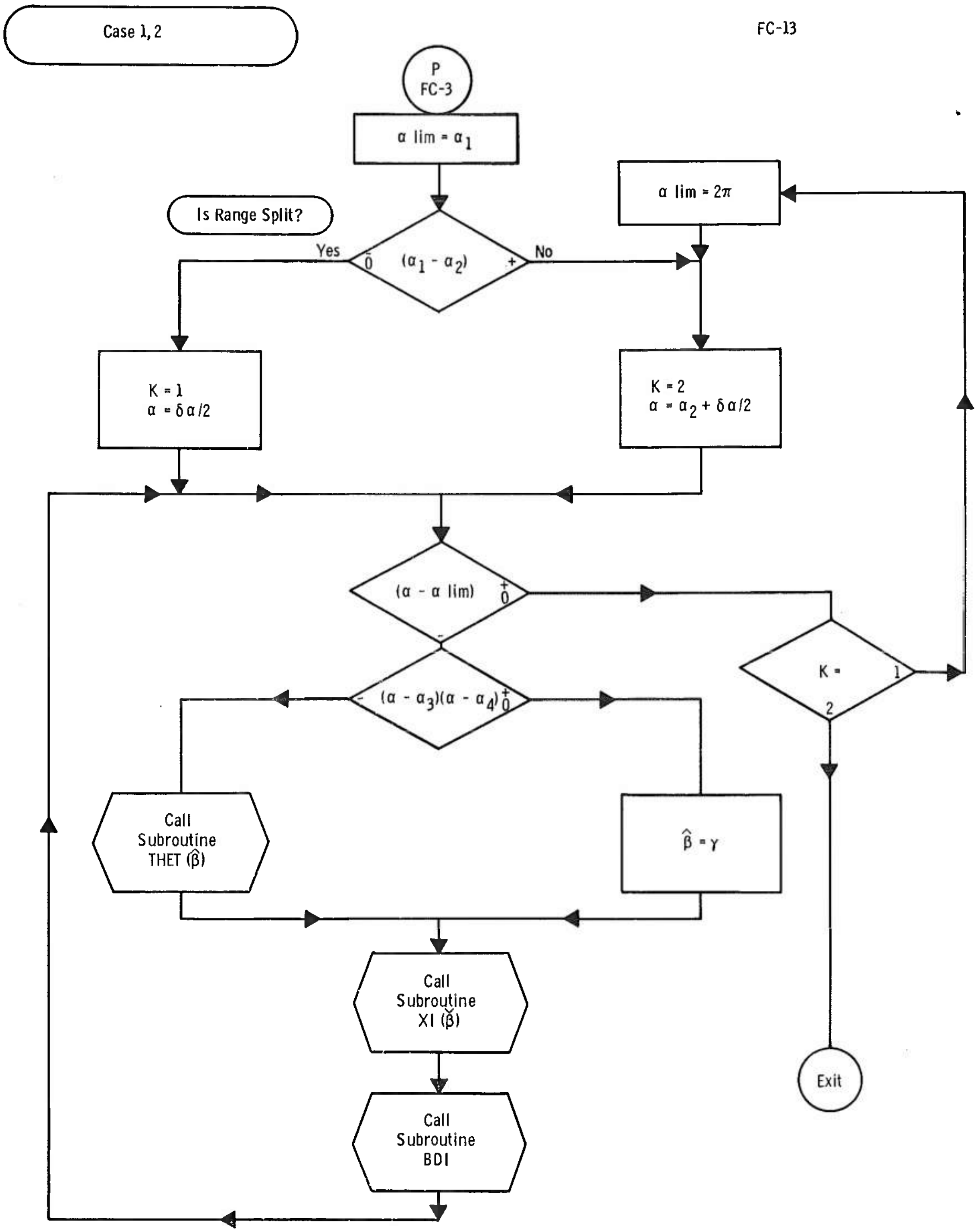


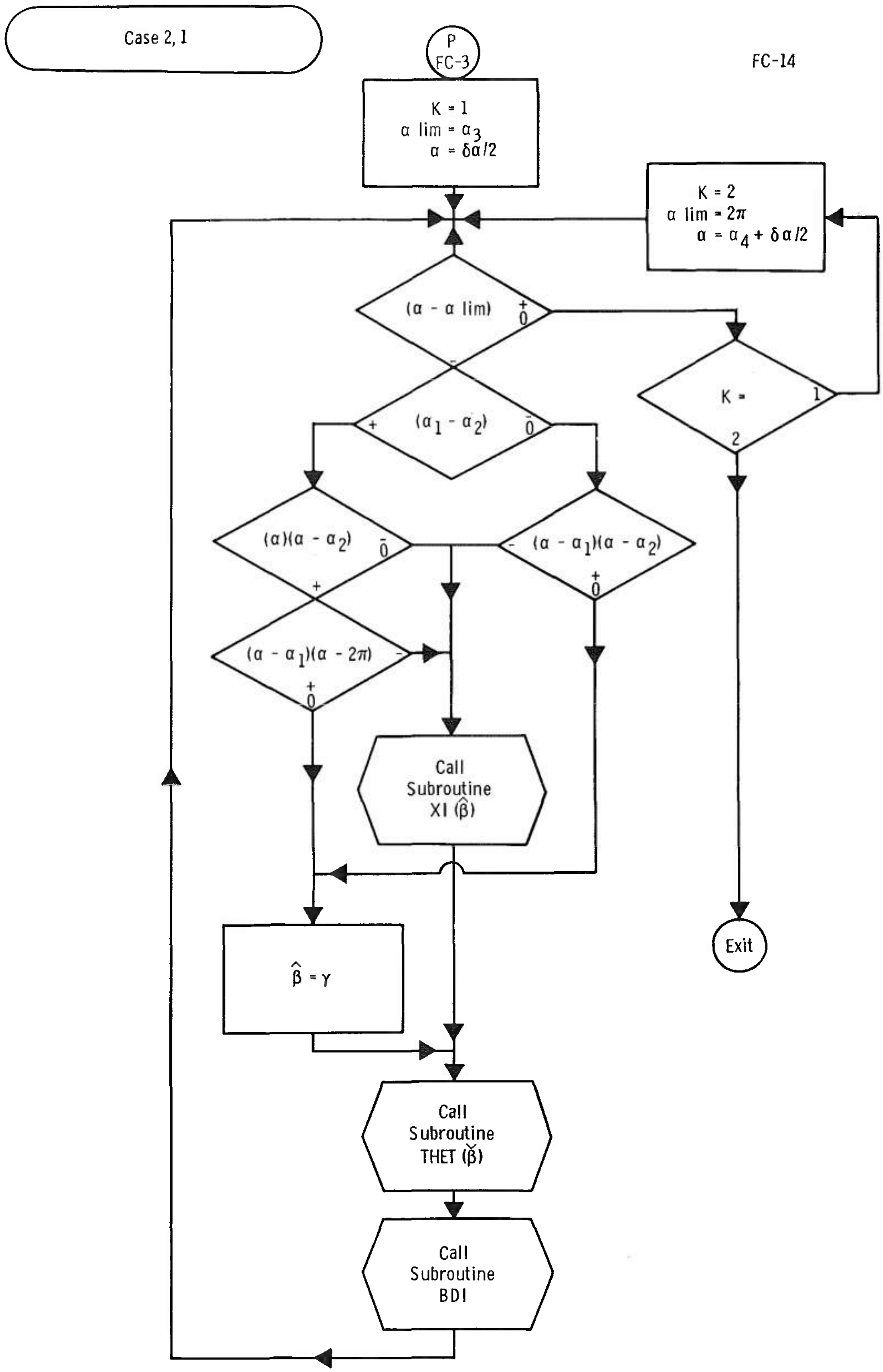
Schematic Only
Details on Next
Three Pages

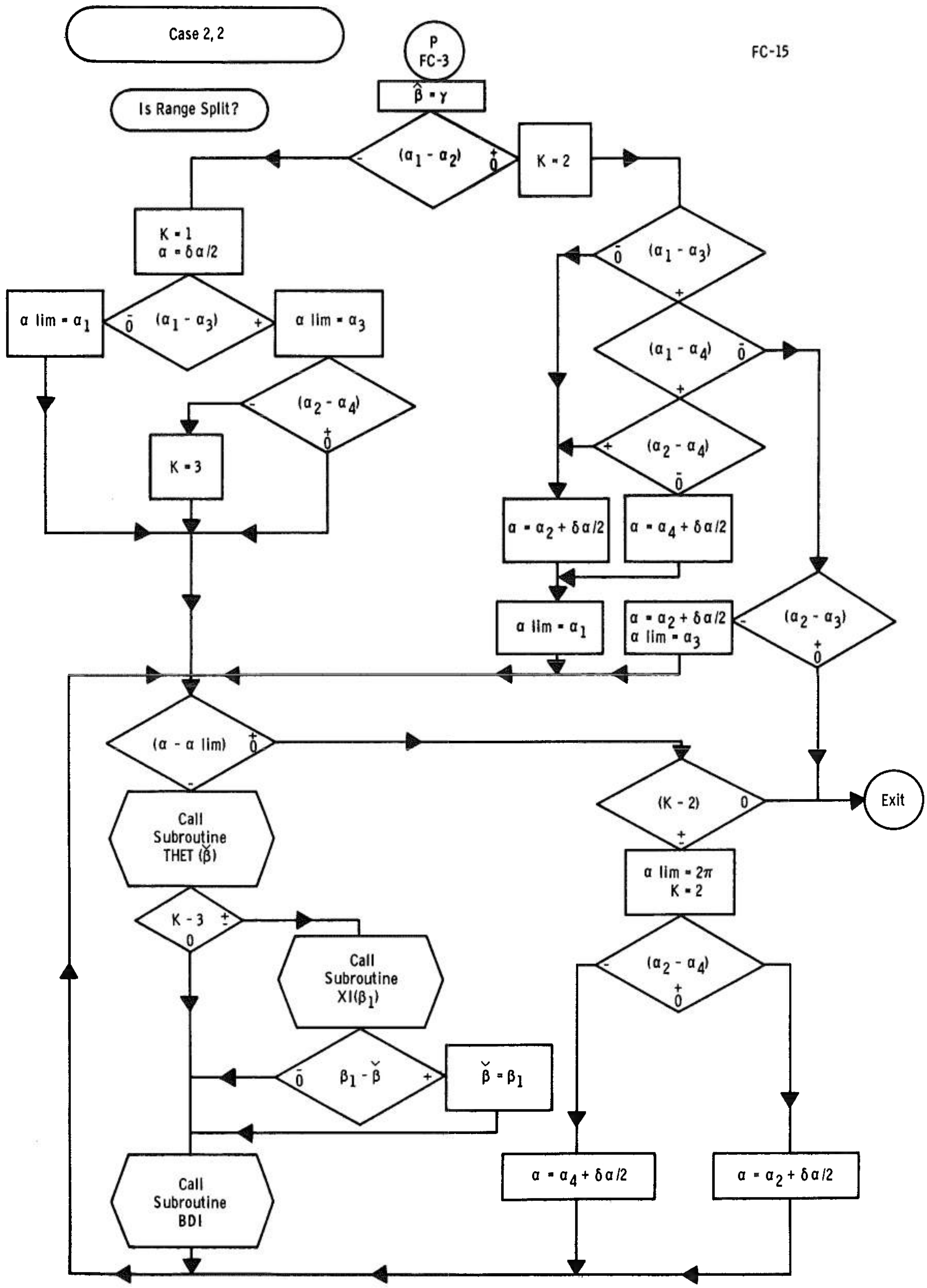

Case I, I
Enter
$\check{\beta} = 0$
$\alpha = \delta\alpha/2$
$\alpha\ \text{lim} = 2\pi$
FC-10
Loop
From
P-
FC-12
$(\alpha - \alpha\ \text{lim})$
Exit
Sort Configurations
$(\alpha_1 - \alpha_2)$
$(\alpha_1 - \alpha_4)$
$(\alpha_1 - \alpha_3)$
$(\alpha_2 - \alpha_3)$
$(\alpha_1 - \alpha_4)$
$(\alpha_2 - \alpha_3)$
$(\alpha_2 - \alpha_4)$
$(\alpha_2 - \alpha_4)$
D
E
H
I
F
G
A
C
B


Case 1, 1
Range Tests
FC-11
D
E
H
From P-FC-10
I
F
G
A
C
B
(α-α3)(α-α1)
(α-α2)(α-α4)
(α-α2)
(α-α3)(α-α2)
(α-α1)(α-α4)
(α-α1)
(α-α1)(α-α4)
(α-α3)(α-α2)
(α-α3)
(α-α2)(α-α1)
(α-α3)(α-α4)
(α-α3)(α-α4)
(α-α1)(α-α2)
(α-α1)(α-α2)
(α-α3)(α-α4)
(α-α2)(α-α4)
(α-α3)(α-α1)
(α-α3)
(α-α3)(α-α2)
(α-α4)(α-α1)
(α-α4)
(α-α1)(α-α4)
(α-α2)(α-α3)
(α-α2)
W
X
Y
Z
To P-FC-12


Case 1, 1
Evaluating
FC-12
From P-FC-11
W
Z
X
Y
K = 1
K = 4
K = 2
K = 3
Call Subroutine THET ($\hat{\beta}$)
Call Subroutine XI ($\hat{\beta}$)
$\hat{\beta} = \gamma$
(K - 4)
-
+
0
$\beta_1 = \hat{\beta}$
(K - 4)
-
+
0
($\hat{\beta} - \beta_1$)
-
0
+
$\hat{\beta} = \beta_1$
Call Subroutine BDI
Loop
To P-FC-10


Case 1, 2
FC-13
P
FC-3
α lim = α1
α lim = 2π
Is Range Split?
Yes
(α1 - α2)
No
K = 1
α = δα/2
K = 2
α = α2 + δα/2
(α - α lim)
K =
1
2
(α - α3)(α - α4)
Call
Subroutine
THET (β̂)
β̂ = γ
Call
Subroutine
XI (β̌)
Call
Subroutine
BDI
Exit


Case 2, 1
P
FC-3
FC-14
K = 1
α lim = α3
α = δα/2
K = 2
α lim = 2π
α = α4 + δα/2
(α - α lim)
K =
1
2
(α1 - α2)
(α)(α - α2)
(α - α1)(α - α2)
(α - α1)(α - 2π)
Call
Subroutine
XI (β̂)
β̂ = γ
Call
Subroutine
THET (β̌)
Call
Subroutine
BDI
Exit



Case 2, 2
Is Range Split?
P
FC-3
FC-15
$\hat{\beta} = \gamma$
$(\alpha_1 - \alpha_2)$
K = 2
K = 1
$\alpha = \delta\alpha/2$
α lim = α_1
$(\alpha_1 - \alpha_3)$
α lim = α_3
$(\alpha_2 - \alpha_4)$
K = 3
$(\alpha_1 - \alpha_3)$
$(\alpha_1 - \alpha_4)$
$(\alpha_2 - \alpha_4)$
$\alpha = \alpha_2 + \delta\alpha/2$
$\alpha = \alpha_4 + \delta\alpha/2$
α lim = α_1
$\alpha = \alpha_2 + \delta\alpha/2$
α lim = α_3
$(\alpha_2 - \alpha_3)$
$(\alpha - \alpha$ lim)
(K - 2)
Exit
Call Subroutine THET $(\check{\beta})$
α lim = 2π
K = 2
K - 3
Call Subroutine XI(β_1)
$(\alpha_2 - \alpha_4)$
$\beta_1 - \check{\beta}$
$\check{\beta} = \beta_1$
Call Subroutine BDI
$\alpha = \alpha_4 + \delta\alpha/2$
$\alpha = \alpha_2 + \delta\alpha/2$

APPENDIX VI
FORTRAN LISTINGS
SUBROUTINE ALBDO
SUBROUTINE BDI
SUBROUTINE XI
SUBROUTINE THET

```
      SUBROUTINE ALBDO(SUM,BETAN,GAM,ALN,THETS,DALS,SGAM,CGAM,STHS,CTHS,
     2 SBN,CBN)
    1 PY = 3.1415927
      PY2= 1.5707963
      EP = .000001
    2 SUM = 0.
C     START THETA SEARCH
      IF(THETS -GAM)400,401,401
  400 AL3 = PY
      AL4 = PY
      K = 1
      GO TO 408
  401 IF(THETS -PY2      )402,402,405
  402 K = 2
      GO TO 407
  405 IF(THETS -PY +GAM )406,575,575
  406 K=3
  407 CAL3 = -SGAM * CTHS/(CGAM * STHS)
      AL3 = PY2 - ASINF(CAL3)
      AL4 = 2.* PY - AL3
C     START BETA SEARCH
  408 IF(BETAN - PY2 + GAM )409,409,410
  409 AL1 = ALN + PY
      AL2 = ALN - PY
      GO TO 417
  410 IF(BETAN - PY2)411,411,414
  411 K=K+3
      GO TO 416
  414 IF(BETAN - PY2 -GAM)415,575,575
  415 K = K+6
  416 CDAL = -CGAM * CBN/ (SGAM * SBN)
      DAL = PY2 - ASINF(CDAL)
      AL1 = ALN + DAL
      AL2 = ALN - DAL
  417 IF(AL1 - 2. * PY)419,419,418
  418 AL1 = AL1 - 2. * PY
  419 IF( AL2 )420,421,421
  420 AL2 = AL2 + 2.* PY
  421 GG=K
      GO TO(422,441,447,425,452,520,434,500,540),K
C                                             START CASE 0-0
  422 ALPHA = .5 * DALS
      BMIN = 0.
      BMAX = GAM
      ALIM = 2. * PY
  423 IF(ALPHA - ALIM )424,575,575
  424 CALL BDI(SUM,ALPHA,DALS,ALN,BMAX,BMIN,SGAM,CGAM,STHS,CTHS,SBN,CBN)
      GO TO 423
C                                             START CASE 0-1
  425 ALPHA =.5 * DALS
      BMIN = 0.
      ALIM = 2.* PY
  426 IF(ALPHA -ALIM)427,575,575
```

```
  427    IF(AL1 - AL2)428,428,429
  428    IF((ALPHA -AL1)*(ALPHA -AL2))432,431,431
  429    IF(ALPHA *(ALPHA -AL2))432,430,430
  430    IF((ALPHA -AL1)*(ALPHA -2.*PY ))432,431,431
  431    BMAX = GAM
         GO TO 433
  432    CALL XI(BMAX,SBN,CBN,ALN,ALPHA)
  433    CALL BDI(SUM,ALPHA,DALS,ALN,BMAX,BMIN,SGAM,CGAM,STHS,CTHS,SBN,CBN)
         GO TO 426
C                                          START CASE 0-2
  434    BMAX = GAM
         ALPHA = AL2 + .5 * DALS
         IF(AL1 -AL2)435,436,436
  435    K = 1
         ALIM = 2. * PY
         GO TO 437
  436    K = 2
         ALIM = AL1
  437    IF(ALPHA - ALIM)440,438,438
  438    IF(K-1)439,439,575
  439    ALPHA = .5 * DALS
         GO TO 436
  440    CALL XI(BMIN,SBN,CBN,ALN,ALPHA)
         CALL BDI(SUM,ALPHA,DALS,ALN,BMAX,BMIN,SGAM,CGAM,STHS,CTHS,SBN,CBN)
         GO TO 437
C                                          START CASE 1-0
  441    ALPHA = .5 * DALS
         BMIN = 0.
         ALIM = 2.* PY
  442    IF(ALPHA - ALIM)443,575,575
  443    IF((ALPHA-AL3) *(ALPHA-AL4))444,445,445
  444    CALL THET(BMAX,STHS,CTHS,SGAM,ALPHA)
         GO TO 446
  445    BMAX = GAM
  446    CALL BDI(SUM,ALPHA,DALS,ALN,BMAX,BMIN,SGAM,CGAM,STHS,CTHS,SBN,CBN)
         GO TO 442
C                                          START CASE 2-0
  447    K=1
         ALPHA = .5 * DALS
         ALIM = AL3
         BMAX = GAM
  448    IF(ALPHA - ALIM)449,450,450
  449    CALL THET(BMIN,STHS,CTHS,SGAM,ALPHA)
         CALL BDI(SUM,ALPHA,DALS,ALN,BMAX,BMIN,SGAM,CGAM,STHS,CTHS,SBN,CBN)
         GO TO 448
  450    IF(K-1)451,451,575
  451    K=2
         ALPHA = AL4 + .5 * DALS
         ALIM = 2.* PY
         GO TO 448
C                                          START CASE 1-1
  452    BMIN = 0.
         ALPHA = .5*DALS
         ALIM = 2.*PY
  453    IF(ALPHA -ALIM)454,575,575
```

```
  454   IF(AL1 -AL2)465,455,455
  455   IF(AL1 -AL4)462,462,456
  456   IF(AL2-AL3)460,460,457
  457   IF((ALPHA -AL3)*(ALPHA -AL1))458,602,602
  458   IF((ALPHA -AL2)*(ALPHA -AL4))601,601,459
  459   IF(ALPHA -AL2)604,603,603
  460   IF((ALPHA -AL2)*(ALPHA -AL1))461,602,602
  461   IF((ALPHA -AL3)*(ALPHA-AL4))601,603,603
  462   IF((ALPHA -AL2)*(ALPHA-AL4))463,602,602
  463   IF((ALPHA -AL3)*(ALPHA -AL1))601,601,464
  464   IF(ALPHA -AL3)603,603,604
  465   IF(AL1 -AL3)476,466,466
  466   IF(AL1 -AL4)467,473,473
  467   IF(AL2 -AL4)471,468,468
  468   IF((ALPHA -AL3)*(ALPHA -AL2))469,603,603
  469   IF((ALPHA -AL1)*(ALPHA -AL4))604,604,470
  470   IF(ALPHA -AL1)601,602,602
  471   IF((ALPHA -AL3)*(ALPHA -AL4))472,603,603
  472   IF((ALPHA -AL1)*(ALPHA -AL2))601,601,604
  473   IF((ALPHA -AL3)*(ALPHA -AL2))474,603,603
  474   IF((ALPHA -AL4)*(ALPHA -AL1))603,603,475
  475   IF(ALPHA -AL4)601,602,602
  476   IF(AL2 -AL3)483,483,477
  477   IF(AL2 -AL4)478,478,481
  478   IF((ALPHA -AL1)*(ALPHA -AL4))479,603,603
  479   IF((ALPHA -AL3)*(ALPHA -AL2))604,480,480
  480   IF(ALPHA -AL3)602,602,601
  481   IF((ALPHA -AL1)*(ALPHA -AL2))482,603,603
  482   IF((ALPHA -AL3)*(ALPHA -AL4))604,602,602
  483   IF((ALPHA -AL1)*(ALPHA -AL4))484,603,603
  484   IF((ALPHA -AL2)*(ALPHA -AL3))603,603,485
  485   IF(ALPHA -AL2)602,602,601
  603   K=3
        BMAX = GAM
        GO TO 491
  601   K =1
        GO TO 486
  604   K =4
  486   CALL THET(BMAX,STHS,CTHS,SGAM,ALPHA)
        IF(K-4)491,487,491
  487   B1= BMAX
        GO TO 488
  602   K=2
  488   CALL XI(BMAX,SBN,CBN,ALN,ALPHA)
        IF(K-4)491,489,491
  489   IF(BMAX -B1)491,491,490
  490   BMAX = B1
  491   CALL BDI(SUM,ALPHA,DALS,ALN,BMAX,BMIN,SGAM,CGAM,STHS,CTHS,SBN,CBN)
        GO TO 453
C                                         START CASE 1-2
  500   ALIM = AL1
        IF(AL1 -AL2)501,501,502
  501   K =1
        ALPHA = .5 *DALS
        GO TO 503
```

```
 502      K =2
          ALPHA =AL2 + .5* DALS
 503      IF(ALPHA -ALIM)506,504,504
 504      IF(K-1)505,505,575
 505      ALIM = 2. * PY
          GO TO 502
 506      IF((ALPHA -AL3)*(ALPHA -AL4))508,507,507
 507      BMAX = GAM
          GO TO 509
 508      CALL THET(BMAX,STHS,CTHS,SGAM,ALPHA)
 509      CALL XI(BMIN,SBN,CBN,ALN,ALPHA)
          CALL BDI(SUM,ALPHA,DALS,ALN,BMAX,BMIN,SGAM,CGAM,STHS,CTHS,SBN,CBN)
          GO TO 503
C                                          START CASE 2-1
 520      K =1
          ALIM =AL3
          ALPHA = .5 * DALS
 521      IF(ALPHA -ALIM)524,522,522
 522      IF(K-1)523,523,575
 523      K =2
          ALIM = 2. * PY
          ALPHA = AL4 + .5*DALS
          GO TO 521
 524      IF(AL1 -AL2)525,525,526
 525      IF((ALPHA -AL1)*(ALPHA -AL2))529,528,528
 526      IF(ALPHA *(ALPHA -AL2))529,529,527
 527      IF((ALPHA- AL1)*(ALPHA -2.*PY))529,528,528
 528      BMAX = GAM
          GO TO 530
 529      CALL XI(BMAX,SBN,CBN,ALN,ALPHA)
 530      CALL THET(BMIN,STHS,CTHS,SGAM,ALPHA)
          CALL BDI(SUM,ALPHA,DALS,ALN,BMAX,BMIN,SGAM,CGAM,STHS,CTHS,SBN,CBN)
          GO TO 521
C                                          START CASE 2-2
 540      BMAX = GAM
          IF(AL1 -AL2)541,545,545
 541      K =1
          ALPHA = .5 *DALS
          IF(AL1 -AL3)542,542,543
 542      ALIM  = AL1
          GO TO 553
 543      ALIM = AL3
          IF(AL2 -AL4)544,553,553
 544      K =3
          GO TO 553
 545      K =2
          IF(AL1 -AL3)551,551,546
 546      IF(AL1- AL4)547,547,549
 547      IF(AL2 -AL3)548,575,575
 548      ALPHA = AL2 +.5* DALS
          ALIM = AL3
          GO TO 553
 549      IF(AL2 -AL4)550,550,551
 550      ALPHA = AL4 +.5* DALS
          GO TO 552
```

```
551     ALPHA = AL2 + .5 *DALS
552     ALIM = AL1
553     IF(ALPHA -ALIM)558,554,554
554     IF(K-2)555,575,555
555     ALIM = 2.* PY
        K =2
        IF(AL2 -AL4)556,557,557
556     ALPHA = AL4 + .5*DALS
        GO TO 553
557     ALPHA = AL2 + .5 *DALS
        GO TO 553
558     CALL THET(BMIN,STHS,CTHS,SGAM,ALPHA)
        IF(K-3)559,561,559
559     CALL XI(B1   ,SBN,CBN,ALN,ALPHA)
        IF(B1 -BMIN)561,561,560
560     BMIN = B1
561     CALL BDI(SUM,ALPHA,DALS,ALN,BMAX,BMIN,SGAM,CGAM,STHS,CTHS,SBN,CBN)
        GO TO 553
575     SUM = SUM * DALS / (SGAM *4. * PY)
        IF(SUM - .00001)576,577,577
576     SUM = 0.
577     BETAN = GG
        RETURN
        END
```

```
      SUBROUTINE BDI(SUM,ALPHA,DALS,ALPHN,BMAX,BMIN,SGAM,CGAM,STHS,CTHS,
     2 SBETN,CBETN)
      IF(BMIN -BMAX)1,2,2
 1    CALPD = COSRF(ALPHA -ALPHN)
      CALPH = COSRF(ALPHA)
      CBMAX = COSRF(BMAX)
      SBMAX = SINRF(BMAX)
      CBMIN = COSRF(BMIN)
      SBMIN = SINRF(BMIN)
      ARG1 = 1.-(SBMAX *SBMAX)/(SGAM * SGAM)
      ARG2 = 1.-(SBMIN *SBMIN)/(SGAM * SGAM)
      IF(ARG1)3,3,4
 3    CPMAX =0.
      GO TO 5
 4    CPMAX = SQRTF(ARG1)
 5    IF(ARG2)6,6,7
 6    CPMIN =0.
      GO TO 8
 7    CPMIN = SQRTF(ARG2)
 8    ARG3 =1. -CPMAX *CPMAX
      ARG4 =1. -CPMIN *CPMIN
      IF(ARG3)9,9,10
 9    PMAX =0.
      GO TO 11
 10   PMAX = ASINF(SQRTF(ARG3))
 11   IF(ARG4)12,12,13
 12   PMIN =0.
      GO TO 14
 13   PMIN = ASINF(SQRTF(ARG4))
 14   FE1 = SBETN* STHS * CALPH * CALPD
      FE2 = CBETN * CTHS
      FE3 = SBETN * CTHS * CALPD
      FE4 = CBETN * STHS * CALPH
      S1 =(2.* FE1 -.5*( FE1 + FE2)* CGAM*CGAM )*  (SGAM* (CBMAX *CPMAX
     1 - CBMIN *CPMIN ) - CGAM *CGAM * LOGF(( CBMAX+ SGAM* CPMAX)/( CBMIN
     2 + SGAM * CPMIN))) - (FE2 + FE1) *SGAM*SGAM*SGAM *(CBMAX* CPMAX*
     3 CPMAX *CPMAX - CBMIN * CPMIN * CPMIN * CPMIN)
      S2 =(FE4 -FE3)* (SBMAX* CPMAX *CPMAX *CPMAX - SBMIN * CPMIN *CPMIN
     1 *CPMIN -.5  * ( SBMAX *CPMAX - SBMIN *CPMIN +SGAM *(PMAX -PMIN)))
     2 *SGAM*SGAM*SGAM
      S3 =(FE2 +FE1)* ( SBMAX**4 - SBMIN**4 )
      S4 = (.5 *FE4+ 1.5 *FE3)*(BMAX -BMIN -SBMAX *CBMAX +SBMIN *CBMIN
     1 ) +(FE4 -FE3)*(SBMAX*SBMAX *SBMAX *CBMAX -SBMIN*SBMIN*SBMIN*CBMIN)
      DSUM = S1 +S2 +S3 +S4
      SUM = SUM + DSUM
 2    ALPHA = ALPHA + DALS
      RETURN
      END
```

```
      SUBROUTINE XI(B,SBN,CBN,AN,AL)
      B = 0.
      TB = -CBN / (SBN * COSRF(AL - AN))
      B =  ATANF(TB)
      RETURN
      END

      SUBROUTINE THET(B,STHS,CTHS,SGAM,AL)
      FE = -CTHS / (STHS * COSRF(AL))
      B = 0.
      B  = ATANF(FE)
      FE = SINRF(B)/(COSRF(B)-SGAM)
      FE = ATANF(FE)
      B = FE -B
      RETURN
      END
```

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13. ABSTRACT

This report is concerned with the development of a fast computer method for evaluating the albedo integral. This integral defines the illumination on an arbitrarily oriented surface element at any point in space about a diffusely reflecting sphere. It enters the calculation of simulation control parameters in the Arnold Engineering Development Center Aerospace Environmental Chamber (Mark I). The seminumerical method developed here is faster than ordinary numerical integration by a factor of about ten. A typical computer program, which formerly required about 30 minutes, now produces the same results in under four minutes.

14. KEY WORDS	LINK A		LINK B		LINK C	
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